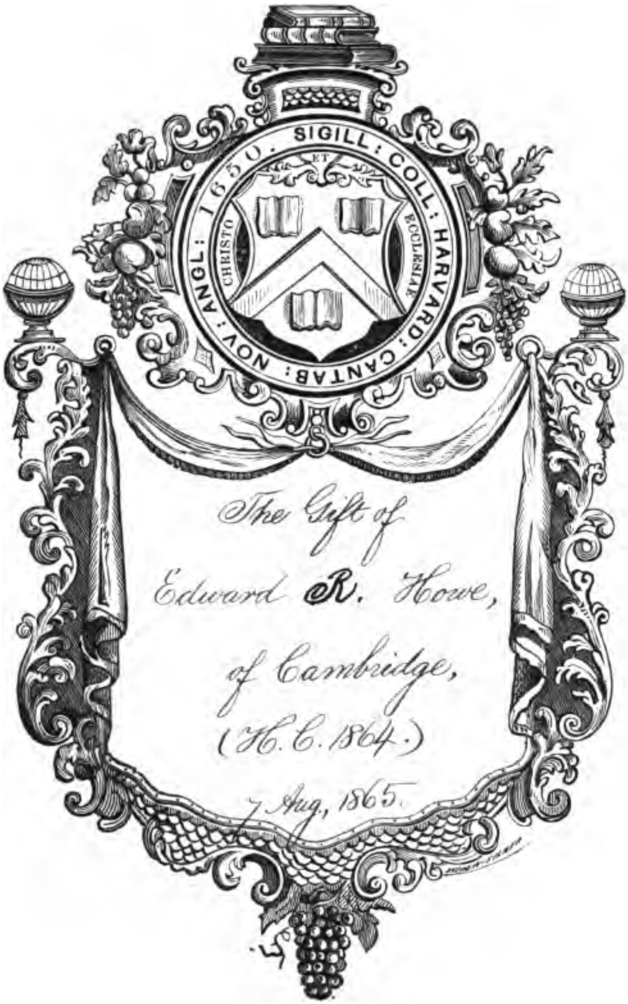
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A

TREATISE

ON

ALGEBRA,

FOR THE USE OF

SCHOOLS AND COLLEGES.

BY

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PREFACE

THE present Treatise is composed substantially of the Elements of Algebra of the author, with such additional matter as fully to adapt it to the advanced course of mathematics now generally pursued in the American Colleges. In its preparation the object has been to give a clear view of the nature and powers of Algebra. The analytic method is uniformly pursued, and the topics are so presented as, in general, to lead the student to feel the want of a new principle before proceeding to its investigation. Thus, the work commences with the exposition of Algebra, as a concise language adapted to facilitate the processes of reasoning required in mathematical investigations. The operations of Algebra, therefore, with which most treatises begin, are not introduced until, in the use of the algebraic language in the solution of questions, the manner in which these operations arise, and the reason for them, are seen. The same general plan is pursued throughout. Much attention is paid to the Discussion of Problems and Equations, a topic of the highest importance to the clear understanding of the true nature of Algebra. A section is given on the Indeterminate Analysis, a subject not usually introduced into our text books, but of great value in itself and in its relation to Analytic Geometry. A full view is given of the General Theory of Equations, and of the method of solving Numerical Equations of any degree. The several subjects are presented in the manner found by experience best adapted to the convenience of recitations and the progress of the pupil. All needed help, it is believed, is furnished, without that diffuseness of explanation which leaves to the learner no room for the exercise of his own powers. The difficulties to be encountered are such only as pertain of necessity to the subject, and which serve to furnish a healthy stimulus to exertion and the mental discipline necessary to the successful prosecution of more advanced studies.

The work in its present form is still better adapted, it is hoped, to the use of Academies and High Schools, in which it has heretofore been extensively used. For a younger class of pupils, the Elementary Algebra of the author will be found sufficiently simple, and an easy introduction to the present work.

WM SMYTH

Bowdoin College, 1853

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ELEMENTS OF ALGEBRA.

SECTION I.—Explanation of Algebraic Signs.

1. Let it be proposed to divide the number 56 into two such parts, that the greater may exceed the less by 12.

To resolve this question, we remark that,

1°. *The greater part is equal to the less added to* 12.

2°. *The greater part, added to the less part, is equal to* 56.

It follows, therefore, that,

3°. *The less part, added to* 12, *added also to the less part, is equal to* 56.

But this language may be abridged, thus,

4°. *Twice the less part, added to* 12, *is equal to* 56; whence,

5°. *Twice the less part is equal to* 56 *diminished by* 12.

Subtracting, therefore, 12 from 56, we have

6°. *Twice the less part equal to* 44; wherefore

7°. *Once the less part is equal to* 44 *divided by* 2, or performing the division, we have

8°. *Once the less part equal to* 22.

Adding 12 to 22 we have 34 for the greater part. The parts required, therefore, are 22 and 34.

2. In the process of reasoning required in the solution of the proposed question expressions, such as "*added to,*" "*diminished by,*" "*equal to,*" &c. are often repeated. These expressions

refer to the operations, by which the numbers given in the question are connected among themselves, or to the relations which they bear to each other. The reasoning, therefore, which pertains to the solution of the proposed, it is evident, may be rendered much more concise, by representing each of these expressions by a convenient sign.

It is agreed among mathematicians to represent the expression "*added to*" by the sign $+$, read *plus*, the expression "*diminished by*" by the sign $-$, read *minus*, the expression "*multiplied by*" by the sign $\times$, that of "*divided by*" by the sign $\div$. Lastly, the expression "*equal to*" is represented by the sign $=$.

3. By means of the above signs, the reasoning in the question proposed may be much abridged; still, however, we have frequent occasion to repeat the expression "*the less part.*" The reasoning, therefore, may be still more abridged by representing this also by a sign.

The less part is the unknown quantity sought directly by the reasoning pursued. It is agreed in general to represent the unknown quantity or quantities sought in a question by some one of the last letters of the alphabet, as, x, y, z.

4. Let us now resume the question proposed, and employ in its solution the signs, which have been explained.

Let us represent by x the less of the two parts required, we have then

$$x + 12 = \text{the greater part,}$$
$$x + 12 + x = 56$$
$$2 \times x + 12 = 56$$
$$2 \times x = 56 - 12$$
$$2 \times x = 44$$
$$x = 44 \div 2$$
$$x = 22.$$

The multiplication of x by 2 may be expressed more concisely thus, $2 . x$, or still more concisely thus, $2x$. Division also is more commonly indicated by writing the number to be divided above a horizontal line, and the divisor beneath it in

the form of a fraction; 14 divided by 2, for example, is indicated thus, $\frac{14}{2}$.

5. The question, which we have solved, is simple; it is sufficient, however, to show the aid which may be derived from convenient signs in facilitating the reasonings, that pertain to the solution of a question. Indeed in abstruse and complicated questions, it would often be difficult, and sometimes absolutely impossible to conduct, without such aid, the reasonings required.

6. The signs which have been explained, together with those which will hereafter be introduced, are called *Algebraic signs*. It is from the use of these that the science of *Algebra* is derived.

Let us now employ the signs already explained in the solution of some questions.

1. Three men, A, B, and C trade in company and gain \$405, of which B has twice as much as A, and C three times as much as B. Required the share of each.

Let x represent the share of A, then $2x$ will represent the share of B and $6x$ the share of C. Then, since the shares added together should be equal to the sum gained, we have

$$x + 2x + 6x = 405$$
$$9x = 405$$
$$x = \frac{405}{9} = 45.$$

Thus we have A's share = \$45; whence B's share is \$90 and C's \$270.

2. A fortress is garrisoned by 2600 men; and there are nine times as many infantry, and three times as many artillery as cavalry. How many are there of each?

3. From two towns, which are 187 miles distant, two travellers set out at the same time, with an intention of meeting. One of them goes 8 miles, and the other 9 miles a day. In how many days will they meet?

4. A gentleman meeting four poor persons distributed 5 shillings among them; to the second he gave twice, to the third thrice, and to the fourth four times as much as to the first. What did he give to each?

5. Four persons, A, B, C and D made a joint stock; B puts in twice as much as A, C puts in three times as much as B, and D puts in as much as the other three together. The whole stock is $20,000. How much did each put in?

6. To divide the number 230 into three such parts, that the excess of the mean above the least may be 40, and the excess of the greatest above the mean may be 60.

Let x represent the least part, then $x+40$ will be the mean and $x+40+60$ will be the greatest part; we have therefore

$$x+x+40+x+40+60=230$$
$$3x+140=230$$
$$3x=90$$
$$x=30.$$

The parts will then be 30, 70 and 130 respectively.

7. A draper bought three pieces of cloth which together measured 159 yards. The second piece was 15 yds. longer than the first, and the third 24 yds. longer than the second. What was the length of each?

8. Three men, A, B and C made a joint stock; A puts in a certain sum, B puts in $115 more than A, and C puts in $235 more that B; the whole stock was $1753. What did each man put in?

9. A gentleman buys 4 horses, for the second of which he gives £12 more than for the first, for the third £6 more than for the second, and for the fourth £2 more than for the third. The sum paid for all was £230. How much did each cost?

10. A man leaves by will his property, amounting to $14000, to his wife, two sons and three daughters; each son is to receive twice as much as a daughter, and the wife as much as all the children together. What will each receive?

11. An express sets out to travel 240 miles in 4 days, but in consequence of the badness of the roads, he found he must go 5 miles the second day, 9 the third and 14 the fourth day less than the first. How many miles must he travel each day?

12. The sum of $300 was divided among 4 persons; the second received three times as much as the first, the third as much as the first and second, and the fourth as much as the second and third. What did each receive?

13. A silversmith has 3 pieces of metal. The second weighs 6 oz. more than twice the first, and the third 9 oz. more than three times the second. The weight of the whole being 52 oz., what is the weight of each?

SECTION II.—Equations.

7. The difference between two numbers is 25 and the greater is 4 times the less; required the numbers.

Let x represent the less, then $x+25$ will represent the greater; but since by the question the greater is four times the less, $4x$ will also represent the greater; these two expressions for the same thing will therefore be equal to each other, and we have

$$x+25=4x.$$

An expression for the equality of two things is called an *equation*. The two equal quantities, of which an equation is composed, are called *members* of the equation; the one on the left of the sign of equality is called the *first* member and the other the *second*.

If a member consists of parts separated by the signs $+$ and $-$, these parts are called *terms*.

Thus in the equation $x+25=4x$, the expression $x+25$ is the first member and $4x$ the second.

The quantities x and 25 are the terms of the first member.

A figure written before a letter, showing how many times the letter is to be taken, is called the *coefficient* of that letter. In the quantities $4x$, $7x$, 4 and 7 are the coefficients of x.

Equations are distinguished into different degrees. An equation, in which the unknown quantity is neither multiplied by itself, nor by any other unknown quantity, is called an equation of *the first degree.*

8. In the solution of a question by the aid of algebraic signs there are, it is evident from the examples already performed, two distinct parts. In the first, we form an equation by means of the relations established by the nature of the question between the known and unknown quantities. This is called *putting the question into an equation.*

In the second part, from the equation, thus formed, we deduce a series of other equations, the last of which gives the value of the unknown quantity. This is called *resolving or reducing the equation.*

9. No general and exact rule can be given for putting a question into an equation When however the equation of a question is formed, there are regular steps for its reduction, which we shall now explain.

Since the two members of an equation are equal quantities, it is evident, that, 1°. *the same quantity may be added to both sides of an equation without destroying the equality;* 2°. *the same quantity may be subtracted from both sides of an equation without destroying the equality;* 3°. *both sides of an equation may be multiplied, or* 4°. *both sides may be divided by the same quantity without destroying the equality.*

10. Let it be proposed to resolve the equation derived from the following enunciation, viz. To find a number such that if one half and one third of this number be added to itself the sum will be equal to 30.

Let x represent the number, then one half of this number will

be represented by $\frac{1}{2}x$ or $\frac{x}{2}$ and one third by $\frac{1}{3}x$ or $\frac{x}{3}$, and we have

$$x+\frac{x}{2}+\frac{x}{3}=30.$$

To resolve this equation we must free the fractional terms from their denominators. In order to this we multiply both sides of the equation first by 2, which gives

$$2x+x+\frac{2x}{3}=60;$$

multiplying next by three, we have

$$6x+3x+2x=180,$$

an equation free from denominators. To free an equation therefore from denominators, *multiply the equation by the denominators successively.*

Ex. 1. Free from denominators the equation

$$\frac{x}{9}+\frac{x}{5}-\frac{x}{7}=49.$$

Ex. 2. Free from denominators the equation

$$\frac{x}{3}+\frac{x}{7}-\frac{x}{12}+\frac{x}{11}=13.$$

Since in this equation the denominator 12 is a multiple of 3, multiplying by 12, we have

$$4x+\frac{12x}{7}-x+\frac{12x}{11}=156.$$

Thus by multiplying first by 12, the number of multiplications necessary to free the equation from denominators is diminished and the equation itself, when freed from denominators, is left in a more simple state.

Ex. 3. Free from denominators the equation

$$\frac{x}{7}+\frac{x}{9}-\frac{x}{21}-\frac{x}{18}=120.$$

Ex. 4. Free from denominators the equation

$$\frac{x}{6}-\frac{x}{4}+\frac{x}{12}-\frac{x}{3}+\frac{x}{2}=10.$$

Ex. 5. Free from denominators the equation

$$\frac{x}{2}+\frac{x}{10}+\frac{3x}{4}-\frac{x}{5}+6=9.$$

The least number divisible by each one of the denominators of the proposed, it is easy to see, is 20. Multiplying by 20, we have

$$10x+2x+15x-4x+120=180;$$

thus the proposed is freed at once from denominators, and the equation which results, it is evident, is the most simple to which it can be reduced free from denominators.

From what has been done, we have the following rule to free an equation from denominators, viz. *Find the least common multiple of the denominators; multiply each term by this common multiple, observing to divide, as we proceed, the numerator of each fractional term by its denominator.*

11. Let it be proposed to resolve the equation

$$3x+25=60-4x.$$

To resolve this equation, it will be necessary to transfer the terms 25 and $4x$ from the members, in which they now stand, to the opposite. In order to this, let us first subtract 25 from both members, we then have

$$3x+25-25=60-4x-25.$$

or $$3x=60-4x-25.$$

Adding next $4x$ to both sides of this last, we have

$$3x+4x=60+4x-4x-25.$$

or $$3x+4x=60-25.$$

Comparing the last equation with the proposed, the term 25 which is additive in the first member has, it is evident, passed into the second member with the sign of subtraction, and the term $4x$ which was subtractive in the second member has passed into the first with the sign of addition. Whence the following rule, for transposing a term from one member of an equation to the other, will be readily inferred, viz. *Efface the term in the member in which it stands, and write it in the other with the contrary sign.*

12. Let it be proposed next to resolve the equation

$$\frac{5x}{12}-\frac{4x}{3}-13=\frac{7}{8}-\frac{13x}{6}.$$

Freeing from denominators, we have

$$10x-32x-312=21-52x;$$

transposing and reducing, we have

$$30x=333;$$

whence dividing both sides by 30 we obtain

$$x=11\tfrac{1}{10}.$$

The unknown quantity in equations of the first degree can be combined with those which are known in four different ways only, viz. by addition, subtraction, multiplication, and division. From what has been done, we have therefore the following rule for the resolution of equations of the first degree with one unknown quantity, viz. 1°. *Free the proposed equation from denominators;* 2°. *bring all the terms, which contain the unknown quantity into the first member and all the known quantities into the other;* 3°. *unite in one term the terms which contain the unknown quantity, and the known quantities in another;* 4°. *divide both sides by the coefficient of the unknown quantity.*

13. Applying the above rule to the equation

$$\frac{x}{6}-\frac{x}{4}+10=\frac{x}{3}-\frac{x}{2}+11, \text{ we obtain } x=12.$$

In order to verify this result we substitute 12 for x in the proposed, it then becomes

$$\frac{12}{6}-\frac{12}{4}+10=\frac{12}{3}-\frac{12}{2}+11;$$

whence performing the operations indicated we obtain

$$9=9.$$

The value $x=12$ satisfies therefore the proposed equation.

In general, to verify the value of the unknown quantity deduced from an equation, we substitute this value for the unknown quantity in the equation. If this renders the two members identically the same, the answer is correct.

14. The following examples will serve as an exercise for the learner in the reduction of equations.

1. $\frac{x}{2}+\frac{x}{3}+\frac{x}{5}=31.$ Ans. $x=30.$

2. $x-7=\frac{x}{5}+\frac{x}{3}$ $x=15.$

3. $\frac{x}{4}+\frac{x}{6}-\frac{x}{10}=\frac{57}{4}.$ $x=45.$

4. $3x+4-\frac{x}{3}=46-2x.$ $x=9.$

5. $\frac{7x}{3}+5x+\frac{2}{3}=28+\frac{5x}{7}-\frac{6}{7}.$ $x=4.$

6. $\frac{3x}{8}-\frac{21}{3}=39-5x+\frac{x}{3}-\frac{5}{8}.$ $x=9.$

7. $\frac{2x}{3}+4=\frac{4x}{5}+12-\frac{5x}{7}.$ $x=13\frac{47}{61}.$

8. $\frac{3x}{5}-\frac{7x}{10}+\frac{3x}{4}=\frac{7x}{8}-15.$ $x=66\frac{2}{3}.$

9. $\frac{x}{6}-\frac{x}{4}+10=\frac{x}{3}-\frac{x}{2}+11.$ $x=12.$

10. $\frac{4x}{5}+\frac{3x}{4}-\frac{7x}{3}=\frac{13x}{10}-\frac{5x}{2}+\frac{5}{4}.$ $x=3.$

The equations above have been taken at random. An equation, however, may always be considered as derived from the enunciation of some question. Thus the first of the above equations may be considered as derived from the following enunciation, viz., *to find a number such that one half, one third, and one fifth of this number may together be equal to* 31.

15. Though no general and exact rule can be given for putting a problem into an equation, yet the following precept will be found very useful for this purpose, viz.: *Indicate by the aid of algebraic signs upon the unknown and known quantities the same reasonings and the same operations, that it would be necessary to perform in order to verify the answer, if it were known.*

Let us illustrate this precept by some examples.

1. A gentleman distributing money wanted 10 shillings to be able to give 5 shillings to each person; he therefore gave each 4 shillings only and found that he had 5 shillings left. Required the number of persons.

In order to verify the answer if it were known, we should multiply it first by 5 and from the product subtract 10; we should next multiply it by 4 and add 5 to the product. The results thus obtained would be equal to each other, if the answer were correct.

Let us indicate the same operations by the aid of algebraic signs. Putting x for the number of persons sought and multiplying x by 5 we have $5x$, subtracting 10 from this we have $5x - 10$; again x multiplied by 4 gives $4x$, adding 5 to this we have $4x + 5$. Then as these two results should be equal we have for the equation of the problem

$$5x - 10 = 4x + 5,$$

which being resolved gives $x = 15$.

2. A person expends the third part of his income in board and lodging, the eighth part in clothes and washing, the tenth part in incidental expenses, and yet saves $318 yearly. What is his yearly income? Ans. $720.

3. A and B. began to play; A with exactly four-ninths the sum B had. After A had won $10, he found that they had each the same sum. What had A at first? Ans. $16.

4. A General having lost a battle found that he had only 3600 men more than half his army left, fit for action; 600 more than one-eighth of his men being wounded, and the rest, which were one-fifth of the whole army, either slain, taken prisoners or missing. Of how many men did his army consist?

Ans. 24,000.

5. A sum of money was to be divided among six poor persons; the second received 10*d.* the third 14*d.* the fourth 25*d.* the fifth 28*d.* and the sixth 33*d.* less than the first. Now the sum distributed was 10*d.* more than the treble of what the first received. What money did the first receive? Ans. 40*d.*

6. A father intends by his will that his three sons should share his property in the following manner. The eldest is to receive 100 pounds less than half the whole property, the second is to receive 80 pounds less than a third of the whole property, and the third is to have 60 pounds less than a fourth of the property. Required the amount of the whole property, and the share of each son.

7. A cistern is supplied by two pipes, the first will fill it alone in three hours, the second in four hours. In what time will the cistern be filled if both run together?

If the time were known, we should verify it by calculating what part of the cistern would be filled by each pipe separately; these parts added together would be equal to the whole cistern. To indicate the same operations by the aid of algebraic signs, let $x =$ the time, and let the capacity of the cistern be represented by 1. It is evident that if one of the pipes will fill the cistern in three hours, in one hour it will fill $\frac{1}{3}$ of it, in x hours it will fill x times as much, that is, a part denoted by $\frac{x}{3}$. In like manner in the time x, the second pipe will fill a part denoted by $\frac{x}{4}$; since then these two parts should be equal to the whole cistern, we have for the equation of the problem

$$\frac{x}{3}+\frac{x}{4}=1,$$

from which we obtain $x = 1\frac{5}{7}$ hours.

8. A cistern is furnished with three cocks, the first will fill it in 5 hours, the second in 13 hours, and by the third it would be emptied in 9 hours. In what time will the cistern be filled if all three run together? Ans. $6\frac{3}{97}$ hours.

9. A gentleman having a piece of work to do hired three men to do it; the first could do it alone in 7 days, the second in 9, the third in 15 days. How long would it take the three together to do it? Ans. $3\frac{12}{101}$ days.

10. To divide the number 247 into three parts, which may be to each other as the numbers 3, 5 and 11.

Two numbers are said to be to each other as 3 to 5, or in proportion of 3 to 5, when the first is three-fifths of the second, or which is the same thing, when the second is five-thirds of the first.

If then one of the parts, the first for example, were known, we should verify it thus. We should find a number, which would be five-thirds of the first part; this would be the second part; we should find also a number which would be eleven-thirds of the first part; this would be the third part; the sum of these parts would then be equal to 247.

To imitate this process let $x =$ the first part, the second will then be $\frac{5x}{3}$ and the third $\frac{11x}{3}$. We have then for the equation of the question

$$x + \frac{5x}{3} + \frac{11x}{3} = 247,$$

whence

$$x = 39.$$

11. A sum of money is to be divided between two persons, A and B, so that as often as A receives 9 pounds, B receives 4. Now it happens that A receives 15 pounds more than B. What are their respective shares? Ans. A £27, B £12.

12. A merchant bought a piece of cloth at the rate of 7 crowns for 5 yards, which he sold again at the rate of 11 crowns for 7 yards, and gained 100 crowns by the traffic. How many yards were there in the piece? Ans. $583\frac{1}{3}$ yds.

13. On an approaching war 594 men are to be raised from three towns A, B, C, in proportion to their population. Now the population of A is to that of B as 3 to 5; whilst the population of B is to that of C as 8 to 7. How many men must each town furnish? Ans. A 144, B 240, C 210.

14. A gentleman employed two workmen at different times, one for 3 shillings, and the other for 5 shillings a day. Now

the number of days added together was 40; and they each received the same sum. How many days was each employed?

If the number of days one of the workmen was employed, the second for example, were known, we should verify it thus, we should subtract this number from 40, this would give the number of days the first workman was employed; multiplying next the number of days the first workman was employed by 3, and that of the second by 5, the two products would be equal.

To indicate the same operations let $x =$ the number of days the second workman was employed, then $40 - x$ will be the number of days the first was employed, and the product of $40 - x$ multiplied by 3 should be equal to $x \times 5$.

The multiplication of $40 - x$ by 3 is indicated by inclosing this quantity in a parenthesis and writing the 3 outside, thus, $3(40 - x)$; we have, therefore, for the equation of the question

$$3\,(40 - x) = 5x.$$

With respect to the multiplication required in this equation, it is evident, since 40 should be diminished by the number of units in x, that 40 multiplied by 3 would be too great for the product required, by the number of units in x multiplied by 3; to obtain the true product therefore from 40×3, we must subtract $x \times 3$; we have then

$$120 - 3x = 5x,$$

from which we obtain $x = 15.$

15. Two workmen received the same sum for their labor; but if one had received 15s. more, and the other 9s. less, then one would have had just three times as much as the other. What did they receive? Ans. 21s.

16. A has three times as much money as B; but if A gains \$50 and B loses \$93, then A will have five times as much money as B. How much has each? Ans. A \$$772\frac{1}{2}$, B \$$257\frac{1}{2}$.

17. A and B engaged in trade, A with £240, and B with £96. A lost twice as much as B, and upon settling their ac-

counts it appeared that A had three times as much remaining as B. How much did each lose? Ans. A £96, B £48.

18. Two merchants engage in trade, each with the same sum; A gains \$150, B loses \$63, when it appears that three times A's money is equal to five times B's. What had each at first? Ans. \$382½.

19. A laborer was hired for 48 days; for each day that he wrought he was to receive 24 shillings, but for each day that he was idle he was to forfeit 12 shillings. At the end of the time he received 504 shillings. How many days did he work and how many was he idle?

To verify the numbers required in this problem we should multiply them, if known, by 24 and 12 respectively; subtracting the last product from the first, the remainder would be 504. To indicate these operations by the aid of algebraic signs let $x =$ the number of days in which the laborer wrought, then $48 - x$ will be the number of days, in which he was idle; $24x$ will be the sum due for the number of days in which he wrought, and $576 - 12x$ will be the sum which he forfeited.

The subtraction of $576 - 12x$ from $24x$ is indicated by inclosing this quantity in a parenthesis and writing the sign $-$ before it, thus, $24x - (576 - 12x)$; we have then for the equation of the question

$$24x - (576 - 12x) = 504.$$

To perform the subtraction required in this equation, it is evident, since 576 should be diminished by $12x$ before subtraction, if we take 576 from $24x$ we subtract too much by $12x$; $12x$ must therefore be added to this result in order to have the true remainder; we have then

$$24x - (576 - 12x) = 24x - 576 + 12x,$$

the equation of the problem therefore becomes

$$24x - 576 + 12x = 504,$$

from which we deduce $$x = 30.$$

20. A father being questioned as to the age of his son replied,

that if from double his present age, the triple of what it was six years ago were subtracted, the remainder would be exactly his present age. Required his age. Ans. 9 years.

21. Divide the number 68 into two such parts, that the difference between 84 and the greater may equal three times the difference between 40 and the less.

Ans. The parts will be 26 and 42.

22. Two men commenced trade; A had twice as much money as B; A gained \$50 and B lost \$90; then if three times B's money be subtracted from A's, four times the remainder will be exactly equal to A's money at first? What had each at first? Ans. A \$$426\frac{2}{3}$, B \$$213\frac{1}{3}$.

23. A person at play won as much as he began with and then lost 18 shillings; after this he lost five-ninths of what remained, and then counting his money, he found he had 14 shillings less than at first. What had he at first?

Let $x =$ the number of shillings he began with, then $2x$ will be the sum he had after winning x, and $2x - 18$ the sum remaining after the first loss, four-ninths of which will be the sum remaining after the second loss. One-ninth of $2x - 18$ is expressed thus, $\frac{2x-18}{9}$,

four-ninths, therefore, will be $\frac{8x-72}{9}$,

and we have for the equation of the question

$$x - \frac{8x-72}{9} = 14;$$

from which we obtain $9x - 8x + 72 = 126$;

whence $x = 54$.

24. Divide the number 96 into two such parts, that four-fifths of the greater, diminished by three-fourths of the less, will be equal to 15. Ans. The parts are $56\frac{4}{31}$ and $39\frac{27}{31}$.

25. It is required to divide 84 into two such parts, that if one-half of the less be subtracted from the greater, and one-

eighth of the greater be subtracted from the less, the remainders shall be equal. Ans. The parts are 48 and 36.

26. A and B began to trade with equal sums of money. In the first year A gained 40 pounds and B lost 40; but in the second A lost one-third of what he then had and B gained a sum less by 40 pounds than twice the sum A had lost; when it appeared that B had twice as much money as A. What money did each begin with? Ans. £32(.

27. What two numbers are as 3 to 5, to each of which if 4 be added the sums will be as 5 to 7?

Let $x =$ the less number, then $\frac{5x}{3} =$ the greater; adding 4 to each, the first will be $x + 4$ and the second

$$\frac{5x}{3} + 4, \text{ or } \frac{5x + 12}{3};$$

but by the question seven-fifths of the first should now be equal to the second, we have therefore

$$\frac{7(x + 4)}{5} = \frac{5x + 12}{3}.$$

28. Divide the number 49 into two such parts, that the greater increased by 6 may be to the less diminished by 11 as 9 to 2.

Ans. The parts are 30 and 19.

29. A and B begin trade, A with triple the stock of B. They gain each \$50, which makes their stocks in the proportion of 7 to 3. Required their original stocks.

Ans. A's \$300, B's 100.

30. A, B and C make a joint stock. A puts in \$60 less than B, and \$68 more than C, and the sum of the shares of A and B is to the sum of the shares of B and C as 5 to 4. What did each put in? Ans A \$140, B \$200, and C \$72.

31. A man being at play lost one fourth of his money and then won 3 shillings; after which he lost one third of what he then had and won 2 shillings; lastly he lost one 7th of what he then had; this being done he had but 12 shillings left. What had he at first? Ans. 20s.

32. There are three pieces of cloth, whose lengths are in the proportion of 3, 5 and 7; and 6 yards being cut off from each, the whole quantity is diminished in the proportion of 20 to 17. Required the length of each piece at first.

Ans. 24, 40 and 56 yds.

33. Two persons, A and B have both the same annual income. A lays by one-fifth of his; but B by spending £80 per annum more than A, at the end of 4 years finds himself £220 in debt. What did each receive and expend annually?

Ans. Their income is £125. A spends £100, B £180.

34. A man bought a horse and chaise for $273. Now if three fourths the price of the horse be subtracted from the price of the chaise, the remainder will be equal to five-elevenths the price of the chaise subtracted from four times the price of the horse. Required the price of each.

SECTION III.—Algebraic Operations.

16. A quantity expressed by algebraic signs is called an *algebraic or literal* quantity. Thus, $a+2b+3x$, ab, xyz, are algebraic or literal quantities.

From what has been done, it is easy to see that we shall have frequent occasion to perform upon algebraic quantities operations analogous to the fundamental operations of arithmetic, viz. addition, subtraction, multiplication and division. The operations upon algebraic quantities, differ however from the corresponding ones in arithmetic in this respect, that the results at which we arrive in the case of algebraic quantities are for the most part only indications of operations to be performed. All that we do is to transform the operations originally indicated into others, which are more simple, or which become necessary in order that the conditions of the question may be fulfilled. Thus, in the equation $x+2x+6x=405$, given by the con-

ditions of question first art. 6, we simplify the operations originally indicated by reducing the expressions $x + 2x + 6x$ to one term, $9x$, by an operation analogous to addition in arithmetic, though not strictly the same. So likewise in question nineteenth, art. 15, though we cannot, strictly speaking, subtract $576 - 12x$ from $24x$, yet, by an operation analogous to subtraction in arithmetic, we indicate upon these quantities operations, which produce the same effect, as the subtraction which the conditions of the question require.

17. Algebraic quantities consisting only of one term are called *monomials*, as $3a, -4b$, &c. Those which consist of two terms are called *binomials*, as $a + b$, $c - d$. Those which consist of three terms are called *trinomials*, &c. In general, quantities consisting of more than one term are called *polynomials*. Quantities consisting only of one term are also called *simple quantities*, and those consisting of more than one term are called *compound quantities*.

Quantities in algebra, which are composed of the same letters, and in which the same letters are repeated the same number of times, are called similar quantities, thus, $3ab$, $7ab$ are similar quantities, so also aab, $5aab$.

ADDITION OF ALGEBRAIC QUANTITIES.

18. 1. Let it be required to add the monomials a, b, c, and d; the result, it is evident, will be $a + b + c + d$.

2. Let the quantities to be added be ab, c, ab, d Here we have as before $ab + c + ab + d$; but the quantities ab, ab in this result are similar, they may therefore be united in one term, thus, $2ab$; whence the sum required will be $2ab + c + d$. To add monomials therefore, *Write them one after the other with the sign + between them, observing to simplify the result by uniting in one, those which are similar.*

3. Let it next be required to add the polynomials $a + b$ and $c + d + e$. The sum total of any number of quantities what-

ever should be equal, it is evident, to the sum of all the parts of which these quantities are separately composed; we have therefore for the sum required $a+b+c+d+e$.

Let the quantities proposed be $a+b$ and $c-d$. If we begin by adding c, the result $a+b+c$ will, it is evident, be too great by the quantity d, since it is not c, which we are to add, but c diminished by d; to obtain the true result, therefore, from $a+b+c$ we must subtract d; whence $c-d$ added to $a+b$ gives

$$a+b+c-d.$$

To add polynomials therefore, *Write in order one after the other the quantities to be added with their proper signs, it being observed that the terms, which have no signs before them, are considered as having the sign* $+$.

19. Let it next be required to add the following quantities.

$$9a+7b-2c$$
$$2a-5c$$
$$8b+c.$$

By the rule just given the sum required will be

$$9a+7b-2c+2a-5c+8b+c.$$

In this result the similar terms $9a$, $2a$ may be united in one, $11a$; also the terms $7b$ and $8b$ give $15b$.

The similar quantities $-2c$, $-5c$ being both subtractive, the effect will be the same, if we unite them in one sum $7c$ and subtract this sum; and as there would still remain the quantity c to be added, instead of first subtracting $7c$ and then adding c to the result, the effect will be the same if we subtract only $6c$.

The sum of the expressions proposed will then be reduced to $11a+15b-6c$.

In order to verify this result, let us put numbers for the letters a, b, c, in the proposed· for example, the numbers, 10, 4, · respectively, and we have

$$
\begin{array}{r}
9a+7b-2c=112 \\
2a-5c=5 \\
8b+c=35 \\
\hline
9a+7b-2c+2a-5c+8b+c=152
\end{array}
$$

Making the same substitution in the expression $11a+15b-6c$, we obtain the same result.

The operation, by which all similar terms are reduced to one, whatever sign they may have, is called *reduction.* To perform this operation, *Take the sum of similar quantities, which have the sign + and that of those which have the sign —; subtract the less of the two sums from the greater and give to the remainder the sign of the greater.*

We have then the following general rule for the addition of algebraic quantities, viz. *Write the quantities in order one after the other with their proper signs, observing to simplify the result by reducing to one, terms which are similar.*

EXAMPLES.

1. To add the quantities

$$
\begin{array}{l}
5x+3y-4z \\
6z+2x-5y+2t \\
3s-4y-2z+x \\
7x-3z+4y-6s \\
\hline
\end{array}
$$

Answer $15x-2y-3z+2t-3s$.

To verify this answer let the numbers 12, 5, 4, 3, 13, be put for the letters x, y, z, t, s, respectively.

2. To add the quantities

$$
\begin{array}{l}
7m+3n-14p+17r \\
3a+9n-11m+2r \\
5p-4m+8n \\
11n-2b-m-r+s \\
\hline
\end{array}
$$

Answer $31n-9m-9p+18r+3a-2b+s$.

3. To add the quantities

$$\begin{array}{l} 11bc + 4ad - 8ac + 5cd \\ 8ac + 7bc - 2ad + 4mn \\ 2cd - 3ab + 5ac + am \\ 9am - 2bc - 2ad + 5cd \\ \hline \end{array}$$

Answer $16bc + 5ac + 12cd + 4mn - 3ab + 10am$.

SUBTRACTION OF ALGEBRAIC QUANTITIES.

20. 1. To subtract a from b. Here the quantities being dissimilar, the subtraction can only be expressed by the sign $-$ thus, $b - a$.

2. To subtract $5a$ from $7a$. The quantities in this case being similar, the subtraction may be performed by means of the coefficients, and the result, it is evident, will be $2a$.

3. To subtract $2b + 3c$ from d. To subtract one quantity from another, we must, it is evident, take from this other the sum of all the parts, of which the quantity to be subtracted is composed. The result required will therefore be

$$d - 2b - 3c.$$

4. To subtract $a - b$ from c. If we begin by subtracting a from c, it is evident, that we shall take away too much by the quantity b, by which a should be diminished before its subtraction; b should therefore be added to $c - a$ to give the true result; whence $a - b$ subtracted from c gives

$$c - a + b.$$

5. To subtract $5c + 3d - 4b$ from $7c - 2d - 5b$. The result, it is easy to see, will be

$$7c - 2d - 5b - 5c - 3d + 4b,$$

which becomes by reduction

$$2c - 5d - b.$$

From what has been done the following rule for the subtraction of algebraic quantities will be readily inferred, viz. *Change*

the signs + into —, and the signs — into + in the quantities to be subtracted, or suppose them to be changed, and then proceed as in addition.

EXAMPLES.

To subtract from $17a + 2m - 9b - 4c + 23d$
the quantity $51a - 27b + 11c - 4d$
Answer $2m - 34a + 18b - 15c + 27d$.

2. To subtract from $5ac - 8ab + 9bc - 4am$
the quantity $8am - 2ab + 11ac - 7cd$
Answer $9bc - 6ac - 6ab - 12am + 7cd$

3. To subtract from
$15abc - 13xy + 21cd - 41x - 25$
the quantity $75xy - 4abc + 16x - 53cd - 31mc$
Answer $19abc - 88xy - 57x + 74cd + 31mc - 25$

MULTIPLICATION OF ALGEBRAIC QUANTITIES.

21. 1. The product of a quantity a by another quantity b is expressed, as we have already seen, thus, $a \times b$, or in a more simple manner, thus, ab. In like manner the product of ab by cd is expressed thus, $ab \times cd$, or thus, $abcd$.

2. The letters a and b are called *factors* of the product ab. So also a, b, c and d are the factors of the product $abcd$. The value of a product, it is easy to see, does not depend at all upon the order, in which its factors are arranged; thus the value of the product arising from the multiplication of a by b will evidently be the same, whether we write ba or ab.

3. Let it be proposed to multiply $3ab$ by $5cd$; by no. 1 we have $3ab\,5cd$, or by no. 2, $3 \times 5abcd$; but the factors 3 and 5 in this result may, it is evident, be reduced to one by multiplying them together; performing this operation, the product required will be $15abcd$. In like manner the product of the quantities $7ab$, $9cd$, $13ef$ will be

$$819abcdef.$$

4. Let it be required to multiply aa by a. According to no. 1 we have for the result aaa; but this expression for the product required may, it is easy to see, be abridged by writing the letter a but once only, and indicating by a figure the number of times this letter enters into it as a factor. The figure which indicates the number of times a given letter enters as a factor in a product is called the *exponent* of that letter. And in order to distinguish the exponent of a letter from a coefficient, we place the exponent at the right hand of the letter and a little above it, the coefficient being always placed before the letter, to which it belongs, and on the same line with it.

According to this method the product aa is expressed by a^2, aaa by a^3, $aaaa$ by a^4, &c.

A letter, which is multiplied once by itself, or which has two for an exponent, is said to be raised to the *second power*. A letter which is multiplied twice successively by itself, or which has 3 for an exponent is said to be raised to the *third* power. In general, the power of a letter is designated according to the figure, which it has for an exponent, thus a with 7 for an exponent is called the *seventh* power of a.

A letter which has no exponent is considered as having unity for its exponent, thus a is the same as a^1.

From what has been said, it will be perceived, that *in order to raise a letter to a given power, it is necessary to multiply it successively by itself as many times less one as there are units in the exponent of this power.*

5. Let it next be required to multiply a^3 by a^5. According to no. 1 the product will be expressed by $a^3\,a^5$. In this product the letter a, it will be observed, occurs three times as a factor, and also five times as a factor, whence on the whole it is found eight times as a factor. The product $a^3\,a^5$ may therefore according to no. 4 be expressed more concisely, thus, a^8. In like manner the product of a^7 by a^9 will be a^{16}. Whence, in general, *The product of two powers of the same letter will have*

for an exponent the sum of the exponents of the multiplier and multiplicand.

6. Let it be proposed next to multiply $a^5 b^2 c$ by $a^4 b^3 c^2 d$. According to no. 1 the product will be $a^5 b^2 c\, a^4 b^3 c^2 d$, or by no. 2, $a^5 a^4 b^2 b^3 c\, c^2 d$; but this expression may be reduced by the rule just given to $a^9 b^5 c^3 d$; whence

$$a^5 b^2 c \times a^4 b^3 c^2 d = a^9 b^5 c^3 d.$$

From what has been done we have the following rule for the multiplication of simple quantities, viz. 1°. *Multiply the coefficients together;* 2°. *write in order in the product thus obtained the letters which are found at once in both the multiplier and multiplicand, observing to give to each letter the sum of the exponents, with which this letter is affected in the two factors;* 3°. *if a letter is found in one of the factors only, write it in the product with the exponent which it has in this factor.*

EXAMPLES.

To multiply	1. $8a^2bc^2$ by $7abcd^2$.	Ans. $56a^3b^2c^3d^2$.
	2. $21a^3b^2cd$ by $8abc^3$.	Ans. $168a^4b^3c^4d$.
	3. $17ab^6c$ by $7df$.	Ans. $119ab^6cdf$.

22. Let us pass to the multiplication of polynomials.

To indicate that a polynomial $a+b$, for example, is multiplied by another $c+d$, we draw a vinculum over each and connect them by the sign of multiplication, thus,

$$\overline{a+b} \times \overline{c+d},$$

or, which is the better method, we inclose each of the quantities in a parenthesis and write them in order one after the other, either with or without a sign of multiplication, thus,

$$(a+b) \times (c+d), \text{ or } (a+b)(c+d).$$

1. To multiply $a+b$ by c. To form the product required, it is evident, that we must take c times each of the parts a and b of which the quantity $a+b$ is composed

The product of	$a+b$
multiplied by	c
is therefore	$ac+bc$.

c*

In like manner	$2a + b^2c + d$
multiplied by	h
gives	$2ah + b^2ch + dh.$

2. To multiply $a - b$ by c. Since $a - b$ is smaller than a by the quantity b, ac the product of a by c, it is evident, will be too large for the product required by b times c or bc; whence to obtain the true result, from ac we must subtract bc.

The product of	$a - b$
multiplied by	c
is therefore	$ac - bc$
In like manner	$a^2 + c^2 - dh - ef$
multiplied by	ah
gives	$a^3h + ahc^2 - adh^2 - ahef.$

From what has been done, it is evident, that, *If two terms each affected with the sign + be multiplied together, the product must have the sign +; but if one of the terms be affected with the sign + and the other with the sign —, the product must have the sign —.*

3. Let it be proposed next to multiply $a - b$ by $c - d$. In this case, it is evident, that, if we take c times $a - b$ the result will be too great by d times $a - b$; whence, to obtain the true product, from c times $a - b$, or, $ac - bc$, we must subtract d times $a - b$ or $ad - bd$,

The product of	$a - b$
multiplied by	$c - d$
is therefore	$ac - bc - ad + bd.$

From this example it appears that, *If two terms be affected each with the sign —, the product of these terms should be affected with the sign +.*

If in the expression of a product there occur similar terms, the expression may be abridged by uniting these terms into one.

$$
\begin{array}{ll}
\text{Thus} & 2ab^2 + a^2 - c^2 \\
\text{multiplied by} & a^2 - ab^2 + c^2 \\
\hline
 & 2a^3b^2 + a^4 - a^2c \\
 & -2a^2b^4 - a^3b^2 + ab^2c^2 \\
 & 2ab^2c^2 + a^2c^2 - c^4 \\
\hline
\text{gives} & a^3b^2 + a^4 - 2a^2b^4 + 3ab^2c^2 - c^4.
\end{array}
$$

To verify this result let $a = 5$, $b = 2$, $c = 3$.

From what has been done we have the following rule for the multiplication of polynomials, viz. 1°. *Multiply each term of the multiplicand by each term of the multiplier, observing with respect to the signs, that if two terms multiplied together have each the same sign, the product must have the sign* $+$, *but if they have different signs, the product must have the sign* $-$; 2°. *Add together the partial products thus obtained, taking care to unite in one, terms which are similar.*

23. A polynomial is said to be *arranged* with reference to some letter, when its terms are written in order according to the powers of this letter. The polynomial

$$a^2b^3 + a^3b - ab^4 + a^4b^2,$$

for example, arranged in descending powers of the letter a stands thus, $a^4b^2 + a^3b + a^2b^3 - ab^4$; arranged in ascending powers of the letter b it stands thus, $a^3b + a^4b^2 + a^2b^3 - ab^4$.

The letter with reference to which the arrangement is made is called the *principal* letter.

To facilitate the multiplication of polynomials, it is usual, 1°. to arrange the quantities to be multiplied according to the powers of the same letter; 2°. to dispose of the partial products in such a manner that those terms, which are similar, shall fall under each other. Let it be proposed, for example, to multiply

$$b^3 + b^2a + a^3 + ba^2 \text{ by } 4b^2 - 3ba + 3a^2.$$

The multiplier and multiplicand being both arranged with reference to the letter a, the work will be as follows:

$$
\begin{array}{l}
a^3 + ba^2 + ab^2 + b^3 \\
3a^2 - 3ba + 4b^2 \\
\hline
3a^5 + 3ba^4 + 3b^2a^3 + 3b^3a^2 \\
\qquad - 3ba^4 - 3b^2a^3 - 3b^3a^2 - 3b^4a \\
\qquad\qquad\qquad 4b^2a^3 + 4b^3a^2 + 4b^4a + 4b^5 \\
\hline
\quad 3a^5 + 4b^2a^3 + 4b^3a^2 + b^4a + 4b^5.
\end{array}
$$

24. The following examples will serve as an exercise in the multiplication of polynomials.

To multiply

1. $5a^3 - 4a^2b + 5ab^2 - 3b^3$

by $4a^2 - 5ab + 2b^2$

Answer $20a^5 - 41a^4b + 50a^3b^2 - 45a^2b^3 + 25ab^4 - 6b^5$

2. $a^3 + 3a^2b + 3ab^2 + b^3$

by $a^3 - 3a^2b + 3ab^2 - b^3$

Answer $a^6 - 3a^4b^2 + 3a^2b^4 - b^6$.

3. $x^4 + x^3y + x^2y^2 + xy^3 + y^4$

by $x - y$

Answer $x^5 - y^5$.

25. A term which contains one literal factor only, is said to be of the *first degree;* a term which contains two literal factors only, is said to be of the *second* degree, &c. In general, *the degree of a term is marked by the number, which expresses the sum of the exponents of the letters, which enter into this term.* The coefficient is not reckoned in estimating the degree of the term. Thus a^2b^3c is a term of the 6th degree, and $7ab^3$ is a term of the fourth degree.

A polynomial is said to be *homogeneous* when all its terms are of the same degree. Thus, $3a^2 - 4ab$, $5a^3 + abc - b^3$ are homogeneous polynomials.

26. From the rules for multiplication, which have been laid down, it follows,

1°. If the polynomials proposed for multiplication are each homogeneous, *the product of these polynomials will also be ho-*

mogeneous, and *the degree of each term of the product will be equal to the sum of the degrees of any two terms whatever of the multiplier and multiplicand.* Thus in the first example, art. 24, all the terms of the multiplicand being of the third degree and those of the multiplier of the second degree, all the terms of the product are of the fifth degree. When therefore the factors of a product are homogeneous, we may readily detect by means of this remark any error in regard to the exponents, which may have occurred in the course of the work.

2°. In the multiplication of polynominals, if there be no reduction of similar terms, *the number of terms in the product will be equal to the number of terms in the multiplicand multiplied by the number of terms in the multiplier.* Thus if there be 5 terms in the multiplicand and 4 in the multiplier, there will be 20 in the product.

3°. But if there be a reduction of similar terms, then the number of terms in the product may be much less. It should be observed, however, that among the different terms of the product there will be two at least, which will not admit of reduction with any other, viz. 1°. *The term arising from the multiplication of the term in the multiplicand affected with the highest exponent of one of the letters, by the term in the multiplier affected with the highest exponent of the same letter.* 2°. *The term arising from the multiplication of the two terms affected with the lowest exponent of the same letter.*

The manner in which an algebraic product is formed by means of its factors is called the *law* of this product. This law, it will readily be perceived, remains always the same, whatever may be the values attributed to the letters which enter into the factors.

27. A product being given, we may sometimes by mere inspection decompose it into its factors, an operation which is frequently useful.

Let there be the product $a^3b - a^2b^2$. In the formation of this product each term, it is evident, has been multiplied by a^2

and also by b, its factors therefore are a^2, b and $a - b$, and it may be put under the form $a^2 b\,(a - b)$.

In like manner the product $ac + ad + bc + bd$ may be put under the form $a\,(c + d) + b\,(c + d)$, or which is the same thing $(a + b)\,(c + d)$.

DIVISION OF ALGEBRAIC QUANTITIES.

28. 1. The object of division in algebra is the same as that of division in arithmetic, viz. *to find one of the factors of a given product, when the other is known.*

According to this definition the divisor multiplied by the quotient must produce anew the dividend; the dividend, therefore, must contain all the factors both of the divisor and quotient; *whence the quotient is obtained by striking out of the dividend the factors of the divisor.*

Thus to divide $abcd$ by ac, we strike out of the dividend the factors a and c of the divisor and obtain bd for the quotient.

2. Let it be required to divide $a^5 b$ by $a^2 b$. Decomposing a^5 into the two factors a^3 and a^2, the dividend may be put under the form $a^3 a^2 b$; whence striking out of the dividend the factors a^2 and b of the divisor, the quotient will be a^3.

From this example it appears that in order to find the quotient of two powers of the same letter; *From the exponent of the dividend we subtract that of the divisor, the remainder will be the exponent of the quotient.*

3. If it be required to divide $72ab^7c$ by $9b^2$, we find that 72, the coefficient of the dividend, may be decomposed into the two factors 9 and 8; b^7 may also be decomposed into the two factors b^5 and b^2; the dividend therefore may be put under the form $9 \times 8ab^5b^2c$; whence, suppressing 9 and b^2, the factors of the divisor, we have $8ab^5c$ for the quotient.

From what has been said we have the following rule for the division of simple quantities, viz. 1°. *Divide the coefficient of*

the dividend by the coefficient of the divisor; 2°. *suppress in the dividend the letters, which are common to it and the divisor, when they have the same exponent, and when the exponent is not the same, subtract the exponent of the divisor from that of the dividend and the remainder will be the exponent to be affixed to the letter in the quotient;* 3°. *write in the quotient the letters of the dividend, which are not in the divisor.*

EXAMPLES.

1. To divide $48a^3b^3c^2d$ by $12ab^2c$. Ans. $4a^2bcd$.
2. To divide $150a^5b^8cd^3$ by $30a^3b^5d^2$. Ans. $5a^2b^3cd$.

29. From the preceding rule, it is evident, in order that the division may be possible, 1°. that the coefficient of the divisor should exactly divide the coefficient of the dividend; 2°. the exponent of a letter in the divisor should not exceed the exponent of the same letter in the dividend; 3°. that there should be no letter in the divisor, which is not found in the dividend.

When these conditions do not exist, the division can only be indicated by the usual sign. If it be required, for example, to divide $12a^2b$ by $9cd$, the division, it is easy to see, cannot be performed; we therefore express the quotient by writing the divisor under the dividend in the form of a fraction, thus, $\frac{12a^2b}{9cd}$.

30. The expression $\frac{12a^2b}{9cd}$ is called an *algebraic fraction.* Fractions of this species may be simplified, in the same manner as those of arithmetic, by striking out the factors, which are common to both terms, or which is the same thing, by dividing both terms by the factors, which are common to them.

Let it be required, for example, to divide $48a^3b^2cd^3$ by $36a^2b^3c^2de$; from what has been said, the most simple expression for the quotient will be $\frac{4ad^2}{3bce}$.

In like manner a^2b divided by $5a^3b$ gives $\frac{1}{5a}$ for the quotient.

31. It sometimes happens, that the exponent of a letter is the same both in the divisor and dividend. The rule for obtaining the exponents of the letters of the quotient, art. 28, being applied to a case of this kind, will give zero for the exponent of the letter in the quotient. Thus, $\frac{a^2}{a^2}$ according to this rule gives a^0 for a quotient; but $\frac{a^2}{a^2}$, it is evident, is equal to unity; the expression a^0 may therefore be considered as equivalent to unity. In general, *a letter with zero for an exponent is to be regarded as a symbol equivalent to unity.*

This symbol, it is evident, will produce no effect upon the value of the expression, in which it appears as a factor, since it signifies nothing but unity. Its only use is to preserve in the work the trace of a letter, which formed a part of the question proposed, but which would otherwise disappear by the effect of division. Thus, if it be required to divide $24a^3b^2$ by $8a^2b^2$, the quotient from what has been said may be put under the form $3ab^0$. The symbol b^0 indicates that the letter b enters 0 times as a factor in this result, or in other words that it does not enter into it as a factor, but at the same time it serves to show that this letter belonged as a factor to the quantities, from which the result $3a$ is obtained by division.

32. We pass next to the division of polynominals. Since the divisor multiplied by the quotient should produce anew the dividend, it is evident, that the dividend must contain all the partial products arising from the multiplication of each term of the divisor, by each term of the quotient. This being the case, it is easy to see, that if we can find any one of these partial products in the dividend, and the particular term of the divisor upon which it depends is known, by dividing this term in the dividend by the known term of the divisor, we shall obtain a term of the quotient sought.

Let it be required to divide

$$50a^3b^2 - 41a^4b + 20a^5 + 10ab^4 - 33a^2b^3$$

by $\quad 5ab^2 - 4a^2b + 5a^3.$

It is evident from what has been said, art. 26, that the term a^5, being affected with the highest exponent of the letter a in the dividend, must have been formed without any reduction from the multiplication of $5a^3$, the term affected with the highest exponent of the letter a in the divisor, by the term affected with the highest exponent of the same letter in the quotient; that is, the term $20a^5$ of the dividend is the product of $5a^3$ of the divisor by a term of the quotient; whence, dividing $20a^5$ by $5a^3$ we obtain $4a^2$ one of the terms of the quotient sought. Multiplying the divisor by $4a^2$, we produce anew all the terms of the dividend, which depend upon $4a^2$, viz. $20a^3b^2 - 16a^4b + 20a^5$; subtracting these from the dividend, the remainder

$$30a^3b^2 - 25a^4b + 10ab^4 - 33a^2b^3$$

must contain all the partial products arising from the multiplication of each one of the remaining terms of the quotient by each term of the divisor.

Regarding this remainder as a new dividend, it is evident, from what has been said, that the term $-25a^4b$ must have arisen from the multiplication of $5a^3$ by the term affected with the highest exponent of the letter a in the remaining terms of the quotient sought; whence, dividing $-25a^4b$ by $5a^3$, we shall be sure to obtain a new term of the quotient.

With regard to the sign, which should be prefixed to this term of the quotient, it is evident, that it should be the sign $-$; since, from the nature of multiplication, the divisor having the sign $+$, the quotient must have the sign $-$ in order that their product may produce anew the dividend $-25a^4b$.

Performing the operation therefore, we have $-5ab$ for another term of the quotient sought. Multiplying the divisor by this term of the quotient, we obtain all the terms of the dividend, which depend upon $-5ab$, viz.

$$-25a^2b^3 + 20a^3b^2 - 25a^4b;$$

subtracting these from $30a^3b^2 - 25a^4b + 10ab^4 - 33a^2b^3$, the remainder $10a^3b^2 + 10ab^4 - 8a^2b^3$, will contain all the partial products arising from the multiplication of each one of the

remaining terms of the quotient sought by each term of the divisor; whence, for the same reasons as before, dividing $10a^3b^2$ by $5a^3$, we have $2b^2$ for a new term of the quotient; multiplying the divisor by this term and subtracting as before, nothing remains; the division is therefore exact, and we have for the quotient sought $4a^2 - 5ab + 2b^2$.

33. In the course of reasoning pursued above, we have been obliged to seek in each of the partial operations the term in the dividend affected with the highest exponent of one of the letters, in order to divide it by the term of the divisor, affected with the highest exponent of the same letter. We avoid this research by arranging the dividend and divisor with reference to the same letter; for, by means of this preparation, the first term at the left of the dividend and the first term at the left of the divisor will, in each of the partial operations, be the two terms which must be divided, one by the other, in order to obtain a term of the quotient.

The following is a table of the calculations in the preceding example, the dividend and divisor being arranged with reference to the letter a, and placed one by the side of the other as in arithmetic.

$$
\begin{array}{l|l}
20a^5 - 41a^4b + 50a^3b^2 - 33a^2b^3 + 10ab^4 & 5a^3 - 4a^2b + 5ab^2 \\
\underline{20a^5 - 16a^4b + 20a^3b^2} & \overline{4a^2 - 5ab + 2b^2} \\
\qquad - 25a^4b + 30a^3b^2 - 33a^2b^3 + 10ab^4 & \\
\qquad \underline{- 25a^4b + 20a^3b^2 - 25a^2b^3} & \\
\qquad\qquad\qquad 10a^3b^2 - 8a^2b^3 + 10ab^4 & \\
\qquad\qquad\qquad \underline{10a^3b^2 - 8a^2b^3 + 10ab^4} &
\end{array}
$$

From what has been done we have the following rule for the division of compound quantities, viz.

Having arranged the divisor and dividend with reference to the powers of the same letter, 1°. *Divide the first term of the dividend by the first term of the divisor, the result will be the first term of the quotient;* 2°. *multiply the whole divisor by the term of the quotient just found, and subtract the result from the dividend;* 3°. *divide the first term of the remainder by the first term*

of the divisor, the result will be the second term of the quotient, 4°. multiply the whole divisor by the second term of the quotient, and subtract the product from the result of the first operation, and continue the same course of operations until all the terms of the dividend are exhausted.

Recollecting, that in multiplication the product of two terms affected with the same sign should have the sign $+$, and that the product of two terms affected with different signs should have the sign $-$, we infer 1°. *that if the two terms of the dividend and divisor have each the same sign, the quotient arising from their division should have the sign $+$; but if they are affected with contrary signs it should have the sign $-$.* This is the rule for the signs.

EXAMPLES.

To divide

1. $a^5+5a^4b+7a^3b^2+3a^2b^3$

by a^3+3ba^2.

2. $x^5+6x^4+9x^3+9x^2+4x+1$

by x^2+x+1.

3. $72x^4-78x^3y-10x^2y^2+17xy^3+3y^4$

by $12x^2-5xy-3y^2$.

4. $x^5-x^4y-13x^3y^2+x^2y^3+12xy^4$

by $x^3+2x^2y-3xy^2$.

34. The dividend and divisor being arranged with reference to the powers of the same letter, if the first term of the dividend is not divisible by the first term of the divisor, we infer that the total division is impossible, or in other words, that there is no polynominal, which multiplied by the divisor will reproduce the dividend; and, in general, we infer that the division cannot be exactly performed, *when the first term of any one of the partial dividends is not divisible by the first term of the divisor.*

When the division cannot be exactly performed, in order to complete the quotient, we write the remainder over the divisor

in the form of a fraction and annex it to the quotient as in arithmetic.

EXAMPLE.

To divide

$5a^7 - 22a^6b + 12a^5b^2 - 6a^4b^3 - 4a^3b^4 + 9a^2b^6$

by $5a^4 - 2a^3b + 4a^2b^2$

$$\text{Answer } a^3 - 4a^2b + 2b^3 + \frac{9a^2b^6 - 8a^2b^5}{5a^4 - 2a^3b + 4a^2b^2}.$$

35. We may remark in passing, that there is some analogy between division in arithmetic and division in algebra with regard to the manner in which the calculations are disposed and performed; there is, however, this essential difference, that in arithmetical division the figures of the quotient are obtained by *trial;* whereas, in algebraic division, we obtain with certainty a term of the quotient sought, by dividing the first term of each partial dividend by the first term of the divisor. In Algebraic division, moreover, we may begin, as it will be easy to see from the remarks, art. 26, at the right instead of the left of the dividend, since, in this case, we shall have merely to operate upon the terms affected with the lowest, instead of those affected with the highest exponents of the letter, in reference to which the arrangement is made; whereas, in arithmetical division, we must always begin at the left. Indeed, such is the independence of the partial operations in algebraic division, that having obtained one of the terms of the quotient and subtracted from the dividend the product of this term by the divisor, we may in the second partial operation divide, one by the other, the two terms of the new dividend and the divisor affected with the highest exponent of any other letter different from that, with reference to which the arrangement is made, and thus obtain a new term of the quotient. It is indeed only for the sake of convenience, that we always regard the same letter in the course of the partial operations necessary to obtain the quotient.

36. In the process of division, the multiplication of the different terms of the quotient by the divisor often produces terms, which are not found in the dividend, and which it is necessary to divide by the first term of the divisor. These terms are such, as cancel each other in the process of forming the dividend by the multiplication of the divisor by the quotient.

To divide $a^3 - b^3$ by $a - b$,

$$\begin{array}{l|l}
a^3 - b^3 & a - b \\
\cline{2-2}
\underline{a^3 - a^2 b} & a^2 + ab + b^2 \\
\quad a^2 b - b^3 & \\
\quad \underline{a^2 b - a b^2} & \\
\qquad a b^2 - b^3 & \\
\qquad \underline{a b^2 - b^3} &
\end{array}$$

If we now multiply the divisor by the quotient in this example, in order to produce anew the dividend, we shall find, that the new terms, which arise in the process of division, are those which cancel each other in the result of multiplication.

EXAMPLES.

1. To divide $6x^4 - 96$ by $3x - 6$.
2. To divide $x^3 + y^3$ by $x + y$.
3. To divide $a^4 - x^4$ by $a - x$.
4. To divide $x^6 - x^4 + x^3 - x^2 + 2x - 1$ by $x^2 + x - 1$.

37. It sometimes happens, that one or both of the quantities, proposed for division, contains several terms affected with the same power of the letter, in reference to which the arrangement is made. The following examples will exhibit the course to be pursued in cases of this kind.

1. To divide
$11a^2b - 19abc + 10a^3 - 15a^2c + 3ab^2 + 15bc^2 - 5b^2c$ by $5a^2 + 3ab - 5bc$.

The terms $11a^2b - 15a^2c$ may be put under the form $(11b - 15c)\,a^2$, or which is the more convenient method

$$\left.\begin{array}{r} 11b \\ -15c \end{array}\right| a^2,$$

a vertical line being employed instead of a parenthesis to indicate that the quantities $11b, -15c$, placed one under the other at the left hand, are multiplied each by a^2. In like manner the terms $-19abc+3ab^2$ may be put under the form $\begin{array}{r|l} -19bc & a \\ +\ 3b^2 & \end{array}$

Arranging the quantities with reference to the letter a, the calculations may be performed as follows.

$$\begin{array}{r|l|l||l} 10a^3+11b & a^2-19bc & a-5b^2c+15bc^2 & 5a^2+3ab-5bc \\ -15c & \quad +3b^2 & & \overline{2a+b} \\ 10a^3+\ 6b & a^2-10bc & a & \quad -3c \\ \hline \end{array}$$

1st Rem.

$$\begin{array}{r|l|l} 5b & a^2-9bc & a-5b^2c+15bc^2 \\ -15c & \quad +3b^2 & \\ 5b & a^2-9bc & a-5b^2c+15bc^2 \\ -15c & \quad +3b^2 & \\ \hline \end{array}$$

2d Rem. 0

Dividing first $10a^3$ by $5a^2$, we have $2a$ for the quotient; subtracting the product of the divisor by $2a$ from the dividend, we obtain the first remainder; dividing the part affected with a^2 in this remainder by $5a^2$, we obtain $b-3c$ for the quotient; multiplying successively each term in the divisor by $b-3c$, we exhaust the dividend; whence the quotient is

$$2a+b-3c.$$

In like manner the following examples may be performed.

2. To divide

$$\begin{array}{r|l|l||l} -a^6-b^2 & a^4+b^4 & a^2+b^6 & \text{by } a^2-b^2-c^2 \\ +2c^2 & \quad -c^4 & \quad +2b^4c^2 & \\ & & \quad +b^2c^4 & \end{array}$$

Answer $\begin{array}{r|l} -a^4-2b^2 & a^2-b^4 \\ +c^2 & \quad -b^2c^2 \end{array}$

3. To divide

$$\begin{array}{r|l|l||l|l} y^2 & x^3-4yzx^2-2y^3 & x+y^2 & \text{by } \quad y & x-y \\ -z^2 & \qquad\qquad +2yz & \quad -z^2 & \quad -z & \quad -z \end{array}$$

Answer $\begin{array}{r|l|l} y & x^2+y & x-y \\ +z & \quad -z & \quad +z \end{array}$

38. When the dividend is not divisible by the divisor, we may still attempt the division, according to the rules which have been given, and continue it at pleasure.

Thus let it be required to divide x by $x+z$.

$$
\begin{array}{l|l}
x & x+z \\
\underline{x+z} & 1-\frac{z}{x}+\frac{z^2}{x^2}-\frac{z^3}{x^3}+\frac{z^4}{x^4}\ \&c. \\
\quad -z & \\
\quad \underline{-z-\frac{z^2}{x}} & \\
\qquad\quad \frac{z^2}{x} & \\
\qquad\quad \underline{\frac{z^2}{x}+\frac{z^3}{x^2}} & \\
\qquad\qquad -\frac{z^3}{x^2} & \\
\qquad\qquad \underline{-\frac{z^3}{x^2}-\frac{z^4}{x^3}} & \\
\qquad\qquad\qquad \frac{z^4}{x^3}\ \&c. &
\end{array}
$$

From the number of terms in the quotient already obtained in the above example, the learner will readily infer a law, by which the quotient may be continued at pleasure without performing any more operations.

39. Miscellaneous examples in the division of algebraic quantities.

1. To divide x^5+1 by $x+1$.
2. To divide $1-5x+10x^2-10x^3+5x^4-x^5$ by $1-2x+x^2$.
3. To divide a^5+x^5 by $a+x$.
4. To divide $a^5-5a^4z+10a^3z^2-10a^2z^3+5az^4-z^5$ by $a^2-2az+z^2$.
5. To divide $6x^6-5x^5y^2+21x^3y^3-6x^4y^4+x^2y^5+15y^6$ by $2x^3-3x^2y^2+5y^3$.

6. To divide 1 by $1 - a$.

7. To divide a by $a - z^2$.

8. To divide $1 + ax + bx^2 + cx^3 + dx^4 +$ &c., by $1 - x$.

$$\text{Ans. } 1 \begin{array}{c|} +1 \\ +a \\ \end{array}\, x \begin{array}{c|} +1 \\ +a \\ +b \end{array}\, x^2 \text{ \&c.}$$

9. To divide $a - bx + cx^2 - dx^3 +$ &c., by $1 + x$.

SECTION IV.—Algebraic Fractions.

40. When the division of two algebraic quantities cannot be exactly performed, the quotient, as we have seen, is expressed in the form of a fraction, the dividend being taken for the numerator and the divisor for the denominator.

A fraction in algebra has the same signification as a fraction in arithmetic; the denominator shows into how many parts unity is divided, and the numerator how many of these parts are taken. Thus, in the algebraic fraction $\frac{a}{b}$, unity is supposed to be divided into b parts, and a number a of these parts is supposed to be taken.

REDUCTION OF FRACTIONS TO THEIR LOWEST TERMS.

41. A fraction is said to be in its lowest terms, when there is no quantity, that will divide both of its terms without a remainder. To reduce a fraction therefore to this state, we suppress in the numerator and denominator the factors, which are common to them.

The suppression of a factor is the same, it is evident, as dividing by the factor, required to be suppressed.

When the two terms of an algebraic fraction are simple quantities, it will be easy, from inspection, to determine the factors

common to them; but if the terms of the fraction are polynomials, this will not be so easy, and we must in this case have recourse to the method of the *greatest common divisor.*

By the greatest common divisor of two algebraic quantities we understand *the greatest in regard to coefficients and exponents, that will exactly divide these quantities.* Its theory rests upon the same two principles, as that of the greatest common divisor in arithmetic, viz.

1°. *The greatest divisor common to two quantities contains as factors all the particular divisors common to these quantities and does not contain any other factors.* 2°. *The greatest divisor common to two quantities is the same with the greatest divisor common to the less of these quantities and the remainder after the division of the greater by the less.*

42. A quantity is said to be *prime* in respect to another quantity, when the two have no factor in common.

From the first of the preceding principles it follows that we may *multiply*, or *divide*, either of the two quantities by any quantity which is prime to the other, without affecting their greatest common divisor; for we shall not, by these operations, either introduce or throw out a factor common to the two quantities; the greatest common divisor, therefore, should remain the same.

This being premised, let it be proposed to find the greatest common divisor of the polynomials

$$a^3 - a^2b + 3ab^2 - 3b^3, \text{ and } a^2 - 5ab + 4b^2.$$

Pursuing the same general course as in arithmetic, we commence by dividing the first of the proposed polynomials by the second; we thus obtain $a + 4b$ for a quotient, with a remainder $19ab^2 - 19b^3$.

By the second of the above principles the question is now reduced to finding the greatest common divisor to this remainder and the divisor $a^2 - 5ab + 4b^2$. But $19ab^2 - 19b^3$ may be put under the form $19b^2(a - b)$; and since the factor $19b^2$ of

this quantity is prime to $a^2-5ab+4b^2$, it may, in virtue of the first of the above principles, be suppressed; thus the question will be still further reduced to finding the greatest common divisor to $a-b$ and $a^2-5ab+4b^2$.

Dividing the last of these two quantities by the first we obtain an exact quotient $a-4b$; whence $a-b$ is their greatest common divisor; and by consequence it is the greatest common divisor of the polynomials proposed.

The following is a table of the calculations.

1st operation
$$\begin{array}{l|l}
a^3-a^2b+3ab^2-3b^3 & a^2-5ab+4b^2 \\
\underline{a^3-5a^2b+4ab^2} & \overline{a+4b} \\
\quad 4a^2b-ab^2-3b^3 & \\
\quad \underline{4a^2b-20ab^2+16b^3} & \\
\qquad\quad 19ab^2-19b^3 & \\
\text{or} \qquad 19b^2(a-b) &
\end{array}$$

2d operation
$$\begin{array}{l|l}
a^2-5ab+4b^2 & a-b \\
\underline{a^2-ab} & \overline{a-4b} \\
\quad -4ab+4b^2 & \\
\quad \underline{-4ab+4b^2} & \\
\qquad 0 &
\end{array}$$

2. To find the greatest common divisor of the polynomials

$$\left.\begin{array}{l} a^3+5a^2b+3ab^2-b^3 \\ a^2+2ab+b^2 \end{array}\right\} \qquad \text{Ans. } a+b.$$

3. To find the greatest common divisor of the polynomials

$$\left.\begin{array}{l} x^3-2x^2y-5xy^2+6y^3 \\ x^2-2xy+y^2 \end{array}\right\} \qquad \text{Ans. } x-y.$$

4. To find the greatest common divisor of the polynomials

$$\left.\begin{array}{l} x^3-3x^2y-10xy^2+24y^3 \\ x^2+4xy+3y^2 \end{array}\right\} \qquad \text{Ans. } x+3y.$$

5. Let it be proposed, next, to find the greatest common divisor of the polynomials

$$5b^3-18b^2a+11ba^2-6a^3, \text{ and } 7b^2-23ba+6a^2.$$

In this example $5b^3$, the first term of the dividend, is not divisible by $7b^2$, the first term of the divisor. It will be observed, however, that 7, the coefficient of the first term of the divisor, will not divide the remaining terms of the divisor. We may,

therefore, in virtue of the first principle, multiply the dividend by 7 without affecting the greatest common divisor sought. Performing this operation, we have for the dividend

$$35b^3 - 126b^2a + 77ba^2 - 42a^3.$$

Dividing next $35b^3$ by $7b^2$, we obtain $5b$ for a quotient. Multiplying the whole divisor by $5b$, and subtracting, we have for a remainder $-11b^2a + 47ba^2 - 42a^3$.

The exponent of b in this remainder, being equal to the exponent of the same letter in the divisor, we continue the operation; and in order to render the first term divisible by the first term of the divisor, we multiply anew by 7, which gives $-77b^2a + 329ba^2 - 294a^3$. Dividing this by the divisor, the quotient is $-11a$, which we separate from the other by a comma, to show that it has no connection with it, and the remainder is $76ba^2 - 228a^3$, or $76a^2(b-3a)$.

Suppressing the factor $76a^2$, the question is reduced to finding the greatest divisor common to $b-3a$ and $7b^2 - 23ba + 6a^2$. Dividing, therefore, the last of these quantities by the first, we obtain an exact quotient $7b-2a$; whence $b-3a$ is the greatest common divisor sought.

See a table of the calculations.

1st operation

$$\begin{array}{r|l}
35b^3 - 126b^2a + 77ba^2 - 42a^3 & 7b^2 - 23ba + 6a^2 \\
\underline{35b^3 - 115b^2a + 30ba^2} & \overline{5b, \qquad -11a} \\
-11b^2a + 47ba^2 - 42a^3 & \\
-77b^2a + 329ba^2 - 294a^3 & \\
\underline{-77b^2a + 253ba^2 - 66a^3} & \\
76ba^2 - 228a^3 & \\
\text{or} \quad 76a^2(b-3a) &
\end{array}$$

2d operation

$$\begin{array}{r|l}
7b^2 - 23ba + 6a^2 & b - 3a \\
\underline{7b^2 - 21ba} & \overline{7b - 2a} \\
-2ba + 6a^2 & \\
\underline{-2ba + 6a^2} & \\
0 &
\end{array}$$

6. To find the greatest common divisor of the polynomials

$$\left.\begin{matrix} a^3 - 10ab^2 + 3b^3 \\ 5a^2 - 16ab + 3b^2 \end{matrix}\right\} \qquad \text{Ans. } a - 3b.$$

$a - 3b$.

7. To find the greatest common divisor of the polynomials

$$\left.\begin{matrix} 3x^3 - 3x^2y + xy^2 - y^3 \\ 4x^2 - 5xy + y^2 \end{matrix}\right\} \qquad \text{Ans. } x - y.$$

The suppression of a factor common to all the terms in the first remainder in the preceding examples, serves not only to simplify the calculations, but is also *indispensable*. Looking at the first example, it is evident that unless the factor $19b^2$ in the first remainder be suppressed, we must multiply all the terms of the new dividend by $19b^2$, in order to render the first term divisible by the first term of the divisor; we should thus introduce into the dividend a factor, which is also contained in the divisor, and by consequence we should introduce into the greatest common divisor sought, a factor, which does not belong to it.

8. Let it be proposed next to find the greatest common divisor of the polynomials

$$15a^5 + 10a^4b + 4a^3b^2 + 6a^2b^3 - 3ab^4$$
$$12a^3b^2 + 38a^2b^3 + 16ab^4 - 10b^5.$$

Before proceeding to the division of the proposed polynomials, we observe that the first contains the letter a as a factor common to all its terms; and since this letter does not enter as a factor into the second polynomial, we may suppress it, as forming no part of the greatest common divisor sought.

For a similar reason, the factor $2b^2$ may be suppressed in the second polynomial. Thus the question is reduced to finding the greatest common divisor of the polynomials

$$15a^4 + 10a^3b + 4a^2b^2 + 6ab^3 - 3b^4$$
$$6a^3 + 19a^2b + 8ab^2 - 5b^3.$$

Pursuing with these polynomials the same course as in the preceding examples, we should multiply the dividend by 6,

the coefficient of the first term of the divisor. But since 15 and 6 have a common factor 3, it will be sufficient to multiply by 2 the other factor of 6, which does not enter into 15; multiplying therefore by 2 and continuing the operations as above, we obtain for the greatest common divisor, $3a^2 + 2ab - b^2$.

9. To find the greatest common divisor of the polynomials

$$\left.\begin{array}{l} 2x^4 - 5x^3 + 6x^2 - 4x + 1 \\ 6x^3 - 5x^2 + 3x - 1 \end{array}\right\}$$

And $x(2x-1)$.
Ans. $2x - 1$.

10. To find the greatest common divisor of the polynomials

$$\left.\begin{array}{l} 6a^2x^3 + 21a^3x^2 - 27a^5 \\ 4x^4 + 5a^2x^2 + 21a^3x \end{array}\right\}$$

Ans. $2x + 3a$.

From what has been done, we have the following rule, by which to find the greatest common divisor of two polynomials, viz. The polynomials proposed being arranged with reference to the same letter, 1°. *We suppress in each the monomial factors which are not found in the other; 2°. we divide one of the polynomials by the other, and if the division cannot be exactly performed, we divide the first divisor by the remainder, and so on, observing to prepare each dividend when necessary in such a manner, as to render the first term divisible by the first term of the divisor, and to suppress in each remainder the monomial factors, which are not contained in the preceding divisor; and that remainder, which will exactly divide the preceding, will be the greatest common divisor sought.*

43. The research for the greatest common divisor of two polynomials admits, in certain cases, of simplifications which we shall now explain.

1. Let it be proposed to find the greatest common divisor of the polynomials

$$5a^6 + 10a^5x + 5a^4x^2$$
$$a^4x + 2a^3x^2 + 2a^2x^3 + ax^4.$$

The letter a, it will be perceived, enters as a factor into each of the terms of the polynomials proposed. This letter will, therefore, be a factor of the greatest common divisor sought.

Suppressing a in the proposed, and applying the rule to the polynomials which result, we obtain $a+x$ for their greatest common divisor. The greatest common divisor sought will, therefore, be $a\,(a+x)$, or a^2+ax.

2. To find the greatest common divisor of the polynomials

$$\left.\begin{array}{l} 9x^4+53x^3-9x^2-18x \\ 5x^3+55x^2+150x \end{array}\right\} \qquad \left\{\begin{array}{l} \text{Ans. } x+5. \\ \text{Ans. } x\,(x+6). \end{array}\right.$$

3. To find the greatest common divisor of the polynomials

$$\left.\begin{array}{l} 6a^4b-10a^3b^2+7a^2b^3-3ab^4 \\ 3a^3b-5a^2b^2+2ab^3 \end{array}\right\} \qquad \left\{\begin{array}{l} \text{Ans. } a-b. \\ \text{Ans. } ab\,(a-b). \end{array}\right.$$

4. Let it be required next to find the greatest common divisor of the polynomials

$$\begin{array}{r|l|l|l} b^2 & a^6+b^3 & a^5+b^4c^2 & a^2 \\ -c^2 & \quad\; -bc^2 & \quad\; -b^2c^4 & \end{array}$$

$$\begin{array}{r|l|l|l} b & a^4+b^2 & a^3+b^3 & a^2 \\ -c & \quad\; -bc & \quad\; -b^2c & \end{array}$$

The proposed, it will readily be perceived, have a simple factor a^2 common to both; recollecting that this will be a factor of the greatest common divisor sought, we suppress it, and the polynomials, which result, will be

$$\begin{array}{r|l|l} b^2 & a^4+b^3 & a^3+b^4c^2, \\ -c^2 & \quad\; -bc^2 & \quad\; -b^2c^4 \end{array} \text{ and } \begin{array}{r|l|l} b & a^2+b^2 & a+b^3. \\ -c & \quad\; -bc & \quad\; -b^2c \end{array}$$

We may now commence the division of one of these polynomials by the other according to the rule, in order to determine their greatest common divisor. Before proceeding to this, however, let us see if there be not a polynomial divisor common to the coefficients of the letter a, with reference to which the arrangement is made.

Comparing for this purpose the two coefficients of the lowest degree b^2-c^2 and $b-c$, we find that $b-c$ will divide both without a remainder. We inquire next if $b-c$ will divide the remaining coefficients of a. This is the case; $b-c$, therefore, is a divisor common to all the coefficients of the two last polynomials. Recollecting that $b-c$ will also be a factor of the

greatest common divisor sought, we suppress $b - c$, and the polynomials, which result, will be

$$\begin{array}{r|l|l} b & a^4 + b^2 & a^3 + b^3c^2 + b^2c^3, \text{ and } a^2 + ba + b^2 \\ +c & \quad\; + bc & \end{array}$$

Applying the rule to these, the first, it will be perceived, contains a factor $b + c$, which is not contained in the second. Suppressing this, it remains to find the greatest common divisor of the polynomials

$$a^4 + ba^3 + b^2c^2, \text{ and } a^2 + ba + b^2.$$

These, it will be found, have no common divisor. The greatest common divisor of the proposed will, therefore, be

$$a^2(b - c), \text{ or } a^2b - a^2c.$$

7. To find the greatest common divisor of the polynomials

$$\begin{array}{r|r|r|r|r|l} x & y^5 - 3x & y^4 - x^2 & y^3 + x^3 & y^2 - x^3 & y \\ -1 & +3 & +3x & -x & +x^2 & \\ & & -2 & & & \end{array}$$

$$\begin{array}{r|r|r|r|l} x^2 & y^4 - 3x^2 & y^3 + x^3 & y^2 - x^3 & y. \\ -1 & +3 & 2x^2 & +x & \\ & & -x & & \\ & & -2 & & \end{array}$$

Ans. $y(y - 1)(x - 1)$.

8. Let it be proposed next to find the greatest common divisor of the polynomials

$$\begin{array}{r|r|r|l} y^3 & x^4 + by^3 & x^3 + bcy^3 & x^3 \\ -yz^2 & +cy^3 & -bcyz^2 & \\ & -byz^2 & & \\ & -cyz^2 & & \end{array}$$

$$\begin{array}{r|r|l} y^2z & x^2 + by^2z & x + bdy^2z \\ -yz^2 & +dy^2z & \quad -bdyz^2 \\ & -byz^2 & \\ & -dyz^2 & \end{array}$$

The simple quantity xy, it will be perceived, will exactly divide each of the terms of the first of the proposed polynomials, and yz those of the second. The factor y common to these quantities will be, it is evident, a factor of the greatest common divisor sought. Setting apart the y therefore as such, and

dividing the first polynomial by xy and the second by yz, the polynomials, which result, will be

$$\begin{array}{r|l|l|l} y^2 & x^2 + by^2 & x^2 + bcy^2 & x \\ -z^2 & \quad + cy^2 & \quad - bcz^2 & \\ & \quad - bz^2 & & \\ & \quad - cz^2 & & \end{array}$$

$$\begin{array}{r|l|l} y & x^2 + by & x + bdy \\ -z & \quad + dy & \quad - bdz \\ & \quad - bz & \\ & \quad - dz & \end{array}$$

The coefficients of the first of these are divisible each by $y^2 - z^2$, and those of the second by $y - z$; but $y - z$, being a factor common to $y^2 - z^2$ and $y - z$, will also, it is evident, be a factor of the greatest common divisor sought; setting it apart, therefore, as such, and dividing the first polynomial by $y^2 - z^2$ and the second by $y - z$, the polynomials, which result, will be

$$\begin{array}{r|l} x^2 + b & x^2 + bcx, \\ + c & \end{array} \text{ and } \begin{array}{r|l} x^2 + b & x + bd \\ + d & \end{array}$$

Applying the rule to these last, we obtain $x + b$ for their greatest common divisor. The greatest common divisor of the proposed will, therefore, be $y\,(y - z)\;(x + b)$.

From what has been done, the following method for finding the greatest common divisor of two polynomials will be readily inferred. viz.

1°. Suppress in the polynomials proposed the greatest simple divisors, which they respectively contain, observing to set aside as a factor of the greatest common divisor sought, the greatest factor, which these divisors have in common. 2°. Suppress in the polynomials, which result, the greatest polynomial divisor, independent of the principal letter, and set aside as a factor of the greatest common divisor sought the greatest factor which these divisors have in common. 3°. Find the greatest common divisor of the polynomials which result, this will be the remaining factor of the greatest common divisor sought, and the product of the several factors, thus obtained, will be the greatest common divisor sought.

44. To reduce a fraction to its lowest terms, we divide the two terms of the fraction by their greatest common divisor.

EXAMPLES.

1. Reduce $\frac{x^4-b^4}{x^5-b^2x^3}$ to its lowest terms. Ans. $\frac{x^2+b^2}{x^3}$.

2. Reduce $\frac{a^4-x^4}{a^3+a^2x-ax^2-x^3}$ to its lowest terms.

Ans. $\frac{a^2+x^2}{a+x}$.

3. Reduce $\frac{6\cancel{a}x^3+\cancel{a}x^2-12\cancel{a}x}{6\cancel{a}x-8\cancel{a}}$ to its most simple form.

Ans. $\frac{2x^2+3x}{2}$.

4. Reduce $\frac{x^3-2x^2-x+2}{x^4-4x^3+6x^2-5x+2}$ to its lowest terms.

Ans. $\frac{x+1}{x^2-x+1}$.

45. Algebraic fractions being of the same nature as fractions in arithmetic, the rules for the fundamental operations are the same. We shall merely subjoin these rules, with some examples under each, the results being reduced to their lowest terms.

MULTIPLICATION OF ALGEBRAIC FRACTIONS.

RULE.—*Multiply the numerators together for a new numerator, and the denominators for a new denominator.*

EXAMPLES.

1. Multiply $\frac{5c}{a^2}$ by $\frac{a^2b^2}{5cx^2}$. Ans. $\frac{b^2}{x^2}$.

2. Multiply $\frac{a^2+2ab+b^2}{cd-d^2}$ by $\frac{d^2}{a+b}$. Ans. $\frac{d(a+b)}{c-d}$.

3. Multiply $\frac{a^2+ax}{a^2-x^2}$ by $\frac{a^3-x^3}{(a-x)^2}$. Ans. $\frac{a^3+a^2x+ax^2}{(a-x)^2}$.

4. Multiply $3x, \frac{x+1}{2a}$ and $\frac{x-1}{a+b}$ together.

Ans. $\frac{3x^3-3x}{2a^2+2ab}$.

5. Multiply $\frac{a^2-x^2}{a+b}$, $\frac{a^2-b^2}{ax+x^2}$ and $a+\frac{ax}{a-x}$ together.

Ans. $\frac{a^2(a-b)}{x}$.

DIVISION OF ALGEBRAIC FRACTIONS.

RULE.—*Invert the divisor, and then proceed as in multiplication.*

EXAMPLES.

1. Divide $\frac{a+b}{x-y}$ by $\frac{x+y}{a-b}$. Ans. $\frac{a^2-b^2}{x^2-y^2}$.
2. Divide $\frac{3ax+x^2}{a^3-x^3}$ by $\frac{x}{a-x}$. Ans. $\frac{3a+x}{a^2+ax+x^2}$.
3. Divide $\frac{x^4-b^4}{x^2-2bx+b^2}$ by $\frac{x^2+bx}{x-b}$. Ans. $\frac{x^2+b^2}{x}$.
4. To divide 12 by $\frac{(a+x)^2}{x}-a$. Ans. $\frac{12x}{a^2+ax+x^2}$.

ADDITION OF ALGEBRAIC FRACTIONS.

RULE.—*Reduce the fractions to a common denominator; then add the numerators together, and place their sum over the common denominator.*

EXAMPLES.

1. Add together $\frac{x+y}{x-y}$ and $\frac{x-y}{x+y}$. Ans. $\frac{2x^2+2y^2}{x^2-y^2}$.
2. Add together $\frac{x^2+y^2}{x+y}$ and $\frac{x^2-y^2}{x-y}$. Ans. $\frac{2(x^2+xy+y^2)}{x+y}$.
3. Add together $\frac{a}{b}$, $\frac{a-3b}{cd}$, and $\frac{a^2-b^2-ab}{bcd}$.

   Ans. $\frac{acd-4b^2+a^2}{bcd}$.
4. Add together $\frac{a^3}{(a+b)^3}$, $-\frac{ab}{(a+b)^2}$, and $\frac{b}{a+b}$.

   Ans. $\frac{a^3+ab^2+b^3}{(a+b)^3}$.

5. Add together $\frac{3a}{(a-2x)^2}$, $\frac{2a+x}{(a+x)(a-2x)}$, and $-\frac{5}{a+x}$.

Ans. $\frac{20ax-22x^2}{(a+x)(a-2x)^2}$.

SUBTRACTION OF ALGEBRAIC QUANTITIES.

RULE.—*Reduce the fractions to a common denominator; then place the difference of their numerators over the denominator, and it will be the difference required.*

EXAMPLES.

1. From $\frac{5x-3}{x+1}$ subtract $\frac{3x+2}{x-1}$. Ans. $\frac{2x^2-13x+1}{x^2-1}$.

2. From $\frac{1}{x-y}$ subtract $\frac{1}{x+y}$. Ans. $\frac{2y}{x^2-y^2}$.

3. From $\frac{az}{a^2-z^2}$ subtract $\frac{a-z}{a+z}$. Ans. $\frac{3az-a^2-z^2}{a^2-z^2}$.

4. From $\frac{2y^2-2y+1}{y^2-y}$ subtract $\frac{y-1}{y}$. Ans. $\frac{y}{y-1}$.

SECTION V.—EQUATIONS OF THE FIRST DEGREE.

46. The rules obtained in the preceding sections, are sufficient for the solution of all equations of the first degree, however complicated. We place below a few examples, involving operations a little more complicated than those, which have been previously introduced.

1. Given $\frac{7x-8}{11}+\frac{15x+8}{13}=3x-\frac{31-x}{2}$, to find the value of x. Ans. $x=9$.

2. Given $\frac{2x+1}{29}-\frac{402-3x}{12}=9-\frac{471-6x}{2}$, to find the value of x. Ans. $x=72$.

3. Given $\frac{9x+20}{36}=\frac{4x-12}{5x-4}+\frac{x}{4}$, to find the value of x.

Ans. $x=8$.

4. Given $\frac{10x+17}{18}-\frac{12x+2}{13x-16}=\frac{5x-4}{9}$, to find the value of x.

Ans. $x=4$.

5. Given $\frac{18x-19}{28}+\frac{11x+21}{6x+14}=\frac{9x+15}{14}$, to find the value of x.

Ans. $x=7$.

PROBLEMS AND EQUATIONS OF THE FIRST DEGREE WITH TWO UNKNOWN QUANTITIES.

47. Most of the questions we have hitherto considered, involve more than one unknown quantity. We have been able to solve them, however, by representing one of the unknown quantities only by a letter, since, by means of this, it has been easy, from the conditions of the question, to express the other unknown quantity. In many questions the solution becomes more simple by representing more than one of the unknown quantities by a letter, and in complicated questions, it is frequently necessary to do this.

The question, art. 1. viz. *To divide the number* 56 *into two such parts that the greater may exceed the less by* 12, presents itself naturally with two unknown quantities. Thus, denoting the less part by x and the greater by y, we have by the conditions of the question

$$x+y=56$$
$$y-x=12.$$

Deducing the value of y from the second equation, we have $y=x+12$; substituting for y in the first equation its value $x+12$, we have $x+x+12=56$, an equation, which contains only one unknown quantity, and from which we obtain $x=22$.

2. A person has two horses and a saddle, which of itself is worth \$10. If the saddle be put upon the first horse, his value will be twice the second; but if the saddle be put upon the second, his value will be three times the first. What is the value of each?

Let $x =$ the value of the first horse, and y that of the second, we have by the question

$$x + 10 = 2y$$
$$y + 10 = 3x.$$

Deducing the value of y from the second of these equations, and substituting it for y in the first, we have

$$x + 10 = 6x - 20;$$

whence $$x = 6.$$

Substituting next for x its value 6 in the second equation, we have $$y + 10 = 18;$$

whence $$y = 8.$$

The process by which one of the unknown quantities in an equation is made to disappear, is called *elimination.* The method of eliminating one of the unknown quantities, pursued above, is called elimination by *substitution.*

48. Since the two members of an equation are equal quantities, it is evident, 1°. *that we may add two equations, member to member, without destroying the equality;* 2°. *we may subtract the members of one equation from those of another without destroying the equality.*

Taking advantage of this remark, we may frequently eliminate one of the unknown quantities in a more simple manner, than by the process of substitution.

Let there be proposed, for example, the equations

$$5x + 7y = 43$$
$$11x + 9y = 69.$$

If either of the unknown quantities in these equations were affected with the same coefficient, we might, it is evident, eliminate this unknown quantity by a simple subtraction. But

if the first equation be multiplied by 9, the coefficient of y in the second, and the second by 7, the coefficient of y in the first, we shall obtain two new equations, which may be substituted for the proposed, and in which the coefficient of y will be equal, viz.

$$45x + 63y = 387$$
$$77x + 63y = 483.$$

Subtracting then the first of these equations from the second, we have $32x = 96$, from which we obtain $x = 3$. Substituting this value of x in either of the proposed we obtain the value of y.

In like manner, if we wish first to eliminate x, we multiply the first of the proposed equations by 11, the coefficient of x in the second, and the second by 5, the coefficient of x in the first; we thus obtain two new equations, which may be substituted for the proposed, and in which the coefficients of x will be equal, viz.

$$55x + 77y = 473$$
$$55x + 45y = 345.$$

Subtracting therefore the second of these equations from the first, we have $32y = 128$; whence $y = 4$.

Let us take as a second example the equations

$$8x - 21y = 33$$
$$6x + 35y = 177.$$

The coefficients of x in these equations have, it will be perceived, a common factor 2. It will be sufficient therefore, in order to render these coefficients equal, to multiply the first equation by 3 and the second by 4. Performing the operations we have

$$24x - 63y = 99$$
$$24x + 140y = 708;$$

whence, subtracting the first of these equations from the second we obtain

$$203y = 609;$$

therefore $$y = 3.$$

In like manner, since the coefficients of y contain the common factor 7, in order to render the coefficients of y equal we multiply the first of the proposed equations by 5 and the second by 3, which gives two new equations,

$$40x - 105y = 165$$
$$18x + 105y = 531;$$

whence by addition we obtain

$$58x = 696;$$

therefore $$x = 12.$$

49. The method of elimination, which we have now explained, is called elimination *by addition and subtraction*, since, the equations being properly prepared, we cause one of the unknown quantities to disappear by addition or subtraction.

In the use of this method, it is important to ascertain whether the coefficients have common factors, since, if this be the case, by omitting the common factors in the multiplications required, the calculations to be performed become more simple. The equations, moreover, should be reduced to the form of the preceding examples,—that is, they should be freed from denominators, the unknown quantities collected each into one term on one side of the sign of equality, and the known quantities collected in one term on the other.

EXAMPLES.

1. To find the values of x and y in the equations

$$4x - 3y = 1$$
$$3x + 4y = 57.$$

2. To find the values of x and y in the equations

$$4x - 9y = 51$$
$$8x + 13y = 191.$$

3. To find the values of x and y in the equations

$$8y - 3x = 29$$
$$6y - 4x = 20.$$

4. To find the values of x and y in the equations

$$\frac{x}{6}+\frac{y}{4}=6$$

$$\frac{x}{4}+\frac{y}{6}=5\tfrac{2}{3}.$$

Ans. $x=12,\ y=16.$

5. To find the values of x and y in the equations

$$\frac{x+2}{3}+8y=31$$

$$\frac{y+5}{4}+10x=192.$$

Ans. $x=19,\ y=3.$

6. To find the values of x and y in the equations

$$\frac{3x-1}{5}+3y-4=15$$

$$\frac{3y-5}{6}+2x-8=7\tfrac{2}{3}.$$

Ans. $x=7,\ y=5.$

7. To find the values of x and y in the equations

$$\frac{7+x}{5}-\frac{2x-y}{4}=3y-5$$

$$\frac{5y-7}{2}+\frac{4x-3}{6}=18-5x.$$

Ans. $x=3,\ y=2$

8. To find the values of x and y in the equations

$$x+1-\frac{3y+4x}{7}=7-\frac{9y+33}{14}$$

$$y-3-\frac{5x-4y}{2}=x-\frac{11y-19}{4}.$$

Ans. $x=6,\ y=5$

50. We pass next to the solution of some questions producing equations involving two unknown quantities.

1. A number consisting of two figures when divided by 4, gives a certain quotient, and a remainder of 3; when divided

by 9 gives another quotient and a remainder of 8. The value of the figure on the left hand is equal to the quotient obtained, when the number was divided by 9, and the other figure is equal to $\frac{1}{17}$ of the quotient obtained, when the number was divided by 4. Required the number.

Let $x =$ the figure in the place of tens, y that in the place of units; then $10x + y =$ the number, and we have by the question

$$\frac{10x+y}{9} = x + \frac{8}{9}, \text{ and } \frac{10x+y}{4} = \frac{3}{4} + 17y.$$

Deducing the values of x and y from these equations, we obtain $x = 7$, $y = 1$. The number required is therefore 71.

2. A purse holds 19 crowns and 6 guineas. Now 4 crowns and 5 guineas fill $\frac{17}{63}$ of it. How many will it hold of each?

Let $x =$ the number of crowns and $y =$ the number of guineas, then $\frac{1}{x} =$ the space occupied by a crown, and $\frac{1}{y} =$ the space occupied by a guinea, we have therefore by the question

$$\frac{19}{x} + \frac{6}{y} = 1, \text{ and } \frac{4}{x} + \frac{5}{y} = \frac{17}{63}.$$

Multiplying the first equation by 5 and the second by 6, subtracting the second from the first and reducing, we obtain $x = 21$, whence $y = 63$.

3. What fraction is that, whose numerator being doubled, and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$? Ans. $\frac{4}{5}$.

4. A owes \$1200, B \$2500; but neither has enough to pay his debts. Lend me, said A to B, the eighth part of your fortune, and I shall be enabled to pay my debts. B answered, I can discharge my debts, if you will lend me the 9th part of yours. What was the fortune of each?

Ans. A's \$900, B's \$2400.

5. A farmer with 28 bushels of barley at 2*s.* 4*d.* per bushel would mix rye at 3 shillings per bushel, and wheat at 4 shillings per bushel, so that the whole mixture may consist of 100 bushels, and be worth 3*s.* 4*d.* per bushel. How many bushels of rye, and how many of wheat must he mix with the barley?

Ans. 20 bushels of rye and 52 bushels of wheat.

6. A and B speculate with different sums; A gains $150, B loses $50, and now A's stock is to B's as 3 to 2. But if A had lost $50 and B gained $100, then A's stock would have been to B's as 5 to 9. What was the stock of each?

Ans. A's $300 and B's $350.

7. A rectangular bowling green having been measured, it was observed, that if it were 5 feet broader, and 4 feet longer, it would contain 116 feet more; but if it were 4 feet broader, and 5 feet longer, it would contain 113 feet more. Required the length and breadth.

Let $x =$ the length, $y =$ the breadth, then $xy =$ the content, and by the first condition $(x + 4)(y + 5) = xy + 116$, &c.

Ans. The length was 12 and the breadth 9 feet.

8. There is a number consisting of two figures, the figure in the place of units being the greater; if the number be divided by the sum of its figures, the quotient is 4; but if the figures be inverted, and the number which results be divided by a number greater by 2 than the difference of the figures, the quotient becomes 14. Required the number. Ans. 48.

9. A person has two horses and two saddles, one of which cost $50, the other $2. If he places the best upon the first horse, and the worst upon the second, then the latter is worth $8 less than the other; but if he puts the worst saddle upon the first, and the best upon the second horse, then the latter is worth $3\frac{1}{4}$ times as much as the former. What is the value of each horse? Ans. The first $30, the second $70.

10. A cistern containing 210 buckets, may be filled by 2

pipes. By an experiment, in which the first was open 4 and the second 5 hours, 90 buckets of water were obtained. By another experiment, when the first was open 7, and the other $3\frac{1}{2}$ hours, 126 buckets were obtained. How many buckets does each pipe discharge in an hour?

Ans. The first 15, and the second 6 buckets.

11. A person having laid out a rectangular bowling green, observed that if each side had been 4 yards longer, the adjacent sides would have been in the proportion of 5 to 4, but if each had been 4 yards shorter, the proportion would have been 4 to 3. What are the lengths of the sides? Ans. 36 and 28 yds.

12. A vintner has two casks of wine, from the greater of which he draws 15 gallons, and from the less 11; and finds the quantities remaining in the proportion of 8 to 3. After the casks become half empty, he puts 10 gallons of water into each, and finds that the quantities of liquor now in them are as 9 to 5. How many gallons will each hold?

Ans. The larger 79 and the smaller 35 gallons.

13. Two persons, A and B, can perform a piece of work in 16 days. They work together for 4 days, when A being called off, B is left to finish it, which he does in 36 days more. In what time would each do it separately?

Ans. A in 24 and B in 48 days.

14. A work is to be printed, so that each page may contain a certain number of lines, and each line a certain number of letters. If we wished each page to contain 3 lines more, and each line 4 letters more, then there would be 224 letters more in each page; but if we wished to have 2 lines less in a page, and 3 letters less in each line, then each page would contain 145 letters less. How many lines are there in each page? and how many letters in each line?

Ans. There are 29 lines in a page and 32 letters in a line.

15. There is a number consisting of two digits, which is

equal to four times the sum of those digits; and if 18 be added to it, the digits will be inverted. What is the number?

Ans. 24.

16. To find a fraction such, that if 3 be subtracted from the numerator and denominator, it is changed into $\frac{1}{4}$, but if 5 be added to the numerator and denominator it becomes $\frac{1}{2}$. What is the fraction? Ans. $\frac{7}{19}$.

17. There is a cistern, into which water is admitted by three cocks, two of which are of exactly the same dimensions. When they are all open, five-twelfths of the cistern is filled in four hours; and if one of the equal cocks be stopped, seven-ninths of the cistern is filled in ten and two-thirds hours. In how many hours would each cock fill the cistern?

Ans. Each of the equal ones in 32 hours and the other in 24.

18. A person owes a certain sum to two creditors. At one time he pays them \$53, giving to one four-elevenths of the sum due to him, and to the other \$3 more than one-sixth of his debt to him. At a second time he pays them \$42, giving to the first three-sevenths of what remains due to him, and to the other one-third of what is due to him. What were the debts? Ans. \$121 and \$36.

PROBLEMS AND EQUATIONS OF THE FIRST DEGREE WITH THREE OR MORE UNKNOWN QUANTITIES.

51. Let now the following question be proposed, viz.

There are three persons, A, B, and C, whose ages are as follows; If from 4 times A's age added to 5 times B's age, we subtract three times C's age, the remainder will be 70; if from 3 times A's age we subtract 4 times B's age, and to the remainder add twice C's age, the sum will be 25; and if twice A's age, 3 times B's, and 5 times C's age be added together, the sum will be 240 What is the age of each?

This question presents itself naturally with three unknown quantities. Thus denoting A's age by x, B's age by y, and C's by z, we have by the question

$$4x + 5y - 3z = 70$$
$$3x - 4y + 2z = 25$$
$$2x + 3y + 5z = 240.$$

Multiplying the first equation by 2, and the second by 3, and adding the results, we obtain

$$17x - 2y = 215.$$

Again, multiplying the second equation by 5, and the third by 2, and subtracting, we obtain,

$$-11x + 26y = 355.$$

We have now two equations with two unknown quantities only. Deducing next the values of x and y from these, in the same manner as in the preceding equations with two unknown quantities, we have $x = 15$, $y = 20$; substituting these values in the first of the proposed equations, we obtain $z = 30$.

52. In the same manner, if there be four equations, with four unknown quantities, we combine the equations two by two, until one of the unknown quantities is eliminated from the whole; we then have three equations with three unknown quantities. Combining next these three, two by two, until one of the unknown quantities is eliminated, we obtain two equations with two unknown quantities, and so on. The process is altogether similar for five or more equations with the same number of unknown quantities.

EXAMPLES.

1. To find the values of x, y, and z in the equations

$$5x - 6y + 4z = 15$$
$$7x + 4y - 3z = 19$$
$$2x + y + 6z = 46.$$

Ans. $x = 3$, $y = 4$, $z = 6$.

2. To find the values of x, y, and z in the equations

$$2x+4y-3z=22$$
$$4x-2y+5z=18$$
$$6x+7y-z=63.$$

Ans. $x=3$, $y=7$, $z=4$.

3. To find the values of x, y and z in the equations

$$3x+5y+7z=179$$
$$8x+3y-2z=64$$
$$5x-y+3z=75.$$

Ans. $x=8$, $y=10$, $z=15$.

4. To find the values of x, y and z in the equations

$$3x+2y-4z=8$$
$$5x-3y+3z=33$$
$$7x+y+5z=65.$$

Ans. $x=6$, $y=3$, $z=4$.

5. To find the values of x, y, z, and u in the equations

$$x+y+z-u=5$$
$$2x+3y-2z+u=2$$
$$5x-2y+z-2u=9$$
$$3x+y-2z+u=2.$$

Ans. $x=2$, $y=1$, $z=3$, $u=1$.

53. It sometimes happens, that all the unknown quantities are not found in each of the equations. In this case, the elimination may, with a little attention, be very readily performed.

1. Let there be proposed, for example, the four following equations, with four unknown quantities, viz.

$$2x-3y+2z=13$$
$$4u-2x=30$$
$$4y+2z=14$$
$$5y+3u=32.$$

With a little examination we see, that the elimination of z from the first and third equations will give an equation in x and

y, and that the elimination of u from the second and fourth equations will also give an equation in x and y. From these last the values of x and y may be readily found. Performing the necessary operations we obtain $x=3$, $y=1$. Substituting next for x its value in the second equation, we have $u=9$, and substituting for y its value in the third, we have $z=5$.

2. To find the values of x, y, z and u in the following equations.

$$\begin{aligned} 3x-\ y+2z&=7\\ 5x+2y-\ u&=5\\ 2x-3y+2z&=2\\ 7y-3u&=2. \end{aligned}$$

Ans. $x=1$, $y=2$, $z=3$, $u=4$.

3. To find the values of the unknown quantities in the following equations

$$\begin{aligned} 5x-\ y-3z&=8\\ 3y-2z+\ t&=0\\ x+2y-\ z&=3\\ 5y-\ u+5t&=5\\ 4y+\ u&=9. \end{aligned}$$

Ans. $x=3$, $y=1$, $z=2$, $u=5$, $t=1$.

54. We pass next to some questions producing three equations with three unknown quantities.

1. Three laborers are employed in a certain work. A and B together can perform it in 8 days, A and C together in 9 days, and B and C together in 10 days. In how many days can each alone perform the same work?

Let x, y and z represent the number of days respectively, then, in one day A will do one-xth part of it, B one-yth part, and C one-zth part of it, and we shall have for the equations of the question,

$$\frac{8}{x}+\frac{8}{y}=1,\ \frac{9}{x}+\frac{9}{z}=1, \text{ and } \frac{10}{y}+\frac{10}{z}=1.$$

Dividing the first of these equations by 8, the second by 9 and the third by 10, we have

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{8},\ \frac{1}{x}+\frac{1}{z}=\frac{1}{9},\ \frac{1}{y}+\frac{1}{z}=\frac{1}{10};$$

subtracting the second equation from the first, and adding the third to the remainder, and reducing, we obtain $y=17\frac{23}{41}$; and in like manner we find $x=14\frac{34}{49}$, $z=23\frac{7}{31}$.

2. It is required to find three numbers, such, that one-half of the first, one-third of the second, and one-fourth of the third shall together make 46; one-third of the first, one-fourth of the second and one-fifth of the third shall together make 35; and one-fourth of the first, one-fifth of the second and one-sixth of the third, shall together make $28\frac{1}{3}$.

Ans. 12, 60, and 80.

3. Three brothers purchased an estate for \$15,000, and the first wanted, in order to complete his part of the payment, half of the property of the second; the second would have paid his share with the help of a third of what the first owned; and the third required, to make the same payment, in addition to what he had, a fourth part of what the first possessed; what was the amount of each one's property?

Ans. \$3,000, \$4,000, and \$4,250, respectively.

4. Three persons, A, B, and C, compare their fortunes. A says to B, give me \$700 of your money, and I shall have twice as much as you retain; B says to C, give me \$1400, and I shall have thrice as much as you have remaining; C says to A, give me \$420, and then I shall have 5 times as much as you retain. How much has each?

Ans. A \$980, B \$1540, C \$2380.

5. Three men, A, B, C, driving their sheep to market, says A to B and C, if each of you will give me 5 of your sheep, I shall have just half as many as both of you will have left. Says B to A and C, if each of you will give me 5 of yours I shall have just as many as both of you will have left. Says C to A and

B, if each of you will give me 5 of yours I shall have just twice as many as both of you will have left. How many had each?

Ans. 10, 20, and 30 respectively.

6. A cistern is furnished with three pipes, A, B and C. By the pipes A and B it can be filled in 12 minutes, by the pipes B and C in 20 minutes, and by A and C in 15 minutes. In what time will each fill the cistern alone, and in what time will it be filled if all three are open together?

Ans. A will fill it in 20, B in 30, and C in 60 minutes, and the three together in 10 minutes.

7. It is required to divide the number 72 into four such parts, that if the first part be increased by 5, the second part diminished by 5, the third part multiplied by 5, and the fourth part divided by 5, the sum, difference, product and quotient shall all be equal. Ans. The parts are 5, 15, 2 and 50.

SECTION VI.—Negative Quantities. Questions producing Negative Results.

55. The length of a certain field is eight rods and the breadth five rods, how much must be added to the length, that the field may contain 30 square rods?

Let $x =$ the quantity to be added, then by the question

$$40 + 5x = 30,$$

and $$5x = 30 - 40,$$

or dividing by 5 $$x = 6 - 8.$$

In this result 8, the quantity to be subtracted, is greater than that, from which it is required to be taken; the subtraction therefore cannot be performed. We may, however, decompose 8 into

two parts 6 and 2, the successive subtraction of which will be equal to that of 8, and we shall have for $6-8$ the equivalent expression, $6-6-2$, which is reduced to $0-2$ or more simply -2, the sign $-$ being retained before the 2 to show that it remains to be subtracted.

A monomial with the sign $-$ prefixed is called a *negative* quantity, thus, -2, $-3a$, $-5ab$, are negative quantities.

Monomials with the sign $+$ either prefixed or understood are called positive quantities. Thus, 2, $3a$, $5ab$ are positive quantities.

Negative quantities, it will be perceived, differ in nothing from positive quantities except in their sign. They are derived from endeavoring to subtract a larger quantity from one that is smaller, and are to be regarded merely as positive quantities to be subtracted.

56. If it now be asked what is the sum of the monomials $+a$, $-b$, $+c$, the question, from what has been said, is reduced to this, what change will be produced in the quantity a, if the quantity b be subtracted from it and the quantity c be added to the remainder. Indicating the operations required to obtain the answer to the question thus proposed, the result will be

$$a-b+c.$$

In order then to add monomials affected with the signs $+$ and $-$, it will be sufficient *to write them one after the other with the signs with which they are affected, when they stand alone.*

57. If we now add the quantities $+b$, $-b$, the result $b-b$, it is evident, will be equal to zero. If then the expression $b-b$ be added to a, it will not affect the value of a; and $a+b-b$ will only be a different form of expression for the same quantity a. If it now be proposed to subtract $+b$ from a, it will be sufficient, it is evident, to efface $+b$ in the equivalent expression $a+b-b$, and the result will be $a-b$. Again, if it be

required to subtract $-b$ from a, it will be sufficient to efface $-b$ in the same expression, and we shall have for the result $a+b$. Thus, to subtract a positive quantity is the same as to add an equal negative quantity, and to subtract a negative quantity is the same as to add an equal positive quantity. To subtract monomials therefore of whatever sign, *we change the signs, and then proceed as in addition.*

58. If we multiply $b-b$ by a, the product must be $ab-ab$, because the multiplicand being equal to zero, the product must be zero. Since then the product of b by a is evidently ab, that of $-b$ by a must be $-ab$, in order that the second term may destroy the first. For a similar reason the product of a by $b-b$ will be $ab-ab$. *Whence if a negative quantity be multiplied by a positive, or a positive by a negative, the product will be negative.*

Again, if we multiply $-a$ by $b-b$, from what has been proved above, the product of $-a$ by b will be $-ab$, the product of $-b$ by $-a$ must therefore be $+ab$, in order that the result may be zero, as it should be, when the multiplier is zero. Whence, *the product of a negative quantity by a negative quantity will be positive.*

The rules for division follow necessarily from those for multiplication. We have therefore the same rules for the signs in the multiplication and division of isolated simple quantities, as are applied to these quantities, when they make a part of polynomials; and in general, *monomials, when they are isolated are combined in the same manner with respect to their signs, as when they make a part of polynomials.*

59. From what has been said, it will be perceived, that the term *addition* does not in algebra, as in arithmetic, always imply augmentation. Thus, the sum of a and $-b$ is, strictly speaking, the difference between a and b; it will therefore be less than a. To distinguish this from an arithmetical sum, we call it an *algebraic sum.* Thus the polynomial

$$3ab-5bc+cd-ef,$$

considered as formed by uniting the quantities

$$3ab, -5bc, +cd, -ef$$

with their respective signs, is called an *algebraic sum.* Its proper acceptation is the arithmetical difference between the sum of the units contained in the terms, which are additive, and the sum of those contained in the terms, which are subtractive.

In like manner the term *subtraction* in algebra does not always imply diminution. Thus $-b$ subtracted from a gives $a+b$, which is greater than a. This result may, however, be called an *algebraic difference,* since it may be put under the form $a-(-b)$.

60. Resuming now the question proposed, art. 55, we have for the answer $x=-2$. In order to interpret this negative result, we return to the equation of the question $40+5x=30$. Here, the addition intended in the enunciation of the question being arithmetical, it is evidently absurd to require that something should be added to 40 in order to make 30, since 40 is already greater than 30. The negative result indicates, therefore, that the question is *arithmetically* impossible, or in other words, that it cannot be solved in the exact sense of the enunciation. If, however, in the equation $40+5x=30$, we substitute -2 for x, we have $40-10=30$, an equation which is exact. In order then that the result may be positive, or which is the same thing, that the question may be *arithmetically* possible, the enunciation should be modified thus,

The length of a certain field is eight rods, and its breadth five rods; how much must be *subtracted* from the length, that the field may contain 30 square rods?

Putting x for the quantity to be subtracted, we have by this new enunciation $40-5x=30$, from which we obtain $x=2$.

2. The length of a certain field is 11 rods and its breadth 7 rods; how much must be subtracted from the length, that the field may contain 98 square rods?

Let $x =$ the quantity to be subtracted; then by the question

$$77 - 7x = 98;$$

whence $$x = -3.$$

To interpret this result, we return to the equation of the question. Here, as an arithmetical subtraction is intended in the enunciation, it is evidently absurd to require, that something should be subtracted from 77 to make 98, since 77 is already less than 98. The question therefore cannot be solved in the exact sense of the enunciation. If, however, instead of x in the equation of the question, we substitute -3, we have $77 + 21 = 98$, an equation which is exact. In order then that the result may be positive, the question should be modified, thus,

The length of a certain field is 11 rods and the breadth 7 rods; how much must be *added* to the length in order that the field may contain 98 square rods?

Resolving the question according to this new enunciation, we obtain $x = 3$.

Let us take as a third example the following question.

3. A laborer wrought for a person 12 days and had his wife and son with him 7 days, and received 46 shillings. He afterwards wrought 8 days, having his wife and son with him 5 days, and received 30 shillings; how much did he earn per day himself, and how much did his wife and son earn?

Let $x =$ the daily wages of the man, and y that of his wife and son; we have by the question

$$12x + 7y = 46$$
$$8x + 5y = 30.$$

Resolving these equations, we obtain $x = 5$, $y = -2$.

In order to interpret this negative result, we substitute 5 for x in the equations above, by which we have

$$60 + 7y = 46$$
$$40 + 5y = 30,$$

equations which are evidently absurd, since it is required to add something to 60 in order to make 46, and to 40 in order to make 30. If, however, we substitute -2 for y in these last we have

$$60-14=46$$
$$40-10=30,$$

equations which are exact. The negative value therefore obtained for y, shows that the allowance made for the wife and son instead of augmenting the pay of the laborer, should be regarded as a charge placed to his account. The question therefore should be modified, thus,

A laborer wrought for a person 12 days and had his wife and son with him 7 days at a certain *expense*, and received 46 shillings. He afterwards wrought 8 days, having his wife and son with him 5 days at expense as before, and received 30 shillings. How much did the laborer earn per day, and how much was charged him per day on account of his wife and son?

Resolving the question, thus stated, we have

$$x=5,\ y=2.$$

61. From what has been done, it will be perceived, that in problems of the first degree, a negative result indicates some inconsistency in the enunciation of the question, arithmetically considered, and at the same time shows how this inconsistency may be reconciled by rendering subtractive certain quantities, which had been regarded as additive, or additive certain quantities, which had been regarded as subtractive.

Negative results, however, in the extended sense, in which the terms addition and subtraction are used in algebra, may be regarded as answers to questions. Thus, in the equation $40+5x=30$; the negative result -2 shows that it is necessary to add -10 to 40 to obtain 30. By means of this extension of the meaning of the terms, addition and subtraction, we may regard as one single question, those, the enunciations of which are such, that the solution, which satisfies one of

the enunciations, will by a mere change of sign satisfy the other also.

62. The following examples will serve as an exercise in the interpretation of negative results.

1. A father is 55 years old, and his son is 16. In how many years will the son be one-fourth as old as the father?

2. What number is that, whose fourth part exceeds its third part by 12?

3. There are two numbers such, that if twice the second be added to the first, the sum will be 20, but if 3 times the second be subtracted from the first, the difference will be 45. What are the numbers?

4. To divide the number 40 into two such parts, that if the first be multiplied by 7 and the second by 5, the sum of the products will be 90.

5. What number must be subtracted from the numbers 70 and 50 respectively in order that their differences may be as 4 to 3?

6. Three persons comparing their property, it is found, that A's and B's together amount to \$1000, A's and C's to \$480, and B's and C's to \$400. What amount of property has each?

7. A laborer wrought for a gentleman 10 days, having his wife with him 5 days and his son 4 days, and received \$14,25. At another time he wrought 8 days, having his wife with him 6 days and his son 3 days, and received \$13. At a third time he wrought 6 days, having his wife with him 4 days and his son 5 days, and received \$8,00. How much did he receive a day himself, and how much for his wife and son severally?

$x = 5.75$ $y = .25$ $z = -5.46$

SECTION VII.—Indeterminate Analysis.

63. Let it be proposed to find two numbers such, that the first added to three times the second shall be equal to 15.

Putting x and y for the numbers sought, we have by the question

$$x + 3y = 15;$$

here as we have two unknown quantities and but one equation, the particular numbers intended in the question proposed cannot be determined. Deducing, however, from the equation the value of one of the unknown quantities, x for example, we have

$$x = 15 - 3y.$$

If we now assume arbitrarily any values whatever for y, we shall obtain values for x, which will satisfy the equation,

thus, let	$y =$	$1,$	$1\frac{1}{2},$	$2,$	$2\frac{1}{3}$	……..
we have	$x =$	$12,$	$10\frac{1}{2},$	$9,$	8	……..
or otherwise	$y =$	$-1,$	$-1\frac{1}{2},$	$-2,$	$-2\frac{1}{3}$	……..
we have	$x =$	$18,$	$19\frac{1}{2},$	$21,$	22	……..

pairs of values for x and y, which, it is easy to see, will satisfy the equation, and the number of which may be increased without limit.

In general, if the conditions of a problem furnish fewer equations, than there are unknown quantities to be determined, the equations of the problem will admit of an infinity of systems of values for the unknown quantities, if we understand by these any quantities whatever, entire or fractional, positive or negative. It is frequently the case, however, that the nature of the question requires, that the values of the unknown quantities should be entire numbers. This circumstance, it is evident, will very much restrict the number of solutions, especially if we reckon the *direct* solutions only, that is to say, solutions in *entire* and *positive* numbers.

Thus, if in the question proposed the numbers sought are

required to be entire and positive, the value of y, it is evident, must not exceed 5; if then we put successively for y

$$y = 0, \quad 1, 2, 3, 4, 5$$

we have

$$x = 15, 12, 9, 6, 3, 0,$$

and the question admits of six different solutions only, the solution in which 0 is reckoned as a value of one of the unknown quantities being included.

Problems of the kind, which we are here considering, are called *indeterminate problems*, and that part of algebra, which relates to the solution of indeterminate problems, is called *indeterminate analysis.*

64. The preceding question, in which the coefficient of one of the unknown quantities is equal to unity, presents no difficulty. We shall now show, that whatever the coefficients of the unknown quantities, the solution of the question proposed may be made to depend upon the solution of an equation, in which the coefficient of one of the unknown quantities is equal to unity.

Let it be proposed then to find the entire values of x and y in the equation $\quad 17x = 542 - 11y.$

Deducing from this equation the value of y, we have

$$y = \frac{542 - 17x}{11};$$

or performing the division as far as possible, we have

$$y = 49 - x + \frac{3 - 6x}{11}.$$

But, by the question the values of x and y should be entire numbers, *it is necessary*, therefore, *and it is sufficient*, that $\frac{3-6x}{11}$ should be equal to an entire number. Let t be this number (t is called an *indeterminate*,) we have

$$y = 49 - x + t$$
$$11t = 3 - 6x. \qquad (2)$$

and the question is now reduced to resolve in entire numbers the equation (2), the coefficients of which are more simple than those of the proposed. Deducing from this equation the value of x and performing the division as far as possible, we have

$$x=-t+\frac{3-5t}{6}.$$

Here, since x and t are entire numbers $\frac{3-5t}{6}$ must be equal to an entire number; let t' be this number, the letter t being marked with an accent to show that it represents a quantity different from that before represented by it, we have

$$x=-t+t'$$

$$6t'=3-5t. \qquad (3)$$

And the question is still further reduced to resolve in entire numbers the equation (3), the coefficients of which are more simple than those of equation (2). Deducing from this equation the value of t, we have

$$t=-t'+\frac{3-t'}{5}.$$

but t and t' in this equation are entire numbers; $\frac{3-t'}{5}$ must therefore be equal to an entire number; let t'' be this number, we have

$$t=-t'+t''$$

$$5t''=3-t'$$

or

$$t'=3-5t'', \qquad (4)$$

and the question is now reduced to resolve in entire numbers the equation (4), in which the coefficient of one of the unknown quantities t' is equal to unity. Indeed, the two principal unknown quantities and the several indeterminates employed are, it is evident, connected together by the equations

$$y=49-x+t$$

$$x=-t+t'$$

$$t=-t'+t''$$

$$t'=3-5t'';$$

if then we give any entire value whatever to t'' and return to the values of x and y corresponding, the values thus found, it is evident, will be entire numbers, and will satisfy the equation proposed. Thus, let $t''=1$, we have $x=-5$, $y=57$, values which, it is easy to see, will satisfy the equation proposed.

To determine with more facility the values of t'' which will give entire values for x and y, we express x and y immediately in terms of t''. In order to this we substitute for t' its value in the equation for t, which gives

$$t=-(3-5t'')+t''=6t''-3.$$

Substituting next for t and t' their values in the equation for x. and for x and t their values in the equation for y, we obtain finally

$$x=6-11t''$$
$$y=40+17t''.$$

If then we make successively $t''=0, 1, 2, 3, \ldots$ or otherwise, $t''=0, -1, -2, -3, \ldots$ in the above, we shall obtain all the entire values of x and y proper to satisfy the equation proposed. But if entire and positive values only are required for x and y, it will be necessary to give to t'' such values only, as will render $6-11t''$, $40+17t''$ positive. It is evident, that $t''=0$, $t''=-1$, $t''=-2$, are the only values of t'', that will fulfil this condition; for, every positive value of t'' will render x negative, and every negative value of t'' numerically greater than 2, will render y negative. Putting therefore $t''=0, -1, -2$ successively, we have

$$x=6,\ \ 17,\ 28$$
$$y=40,\ 23,\ \ 6.$$

The proposed therefore admits of three different solutions in entire and positive numbers, and of three only.

2. Let it be proposed, as a second example, to divide the number 159 into two such parts, that the first may be divisible by 8, and the second by 13.

Designating by x and y the quotients, arising from the division

of the parts sought by the numbers 8 and 13 respectively, we have by the question $8x + 13y = 159$.

Pursuing with this equation the same process, as in the preceding example, we have the five equations

$$\begin{aligned} x &= 19 - y + t \\ y &= 1 - t + t' \\ t &= \quad - t' + t'' \\ t' &= 1 - t'' - t''' \\ t'' &= 2t'''. \end{aligned}$$

Expressing x and y in terms of t''', we have

$$\begin{aligned} x &= 15 + 13t''' \\ y &= 3 - 8t'''. \end{aligned}$$

Here it is evident, that $t''' = 0$, and $t''' = -1$ are the only values of t''', which will give entire and positive values for x and y. Making successively $t''' = 0$, $t''' = -1$, we have

$$\begin{aligned} x &= 15, 2 \\ y &= 3, 11. \end{aligned}$$

Since then $8x$ and $13y$ represent the parts required, the proposed admits of two solutions, viz. 120 and 39 for the first solution, and 16 and 143 for the second.

The expressions, $x = 15 + 13t'''$, $y = 3 - 8t'''$, are called *formulas* for x and y, since they indicate the manner in which the values of x and y are obtained. In the use of these formulas the accents, it is evident, may be omitted, as we have now no further occasion to distinguish, one from the other, the indeterminates which have been employed.

3. It is required to divide 25 into two parts, one of which may be divisible by 2, and the other by 3.

Ans. The parts are 16 and 9, 10 and 15, or 4 and 21.

4. A person has in his pocket pieces of 5 shillings and 3 shillings only, and wishes to pay a bill of 58 shillings. How many pieces must he give of each?

Ans. 2 of the first and 16 of the second, or &c.

5. A sum of \$81 was distributed among some poor persons, men and women; each woman received \$5, and each man \$7. How many men and women were there?

Ans. There were 3 men and 12 women, or &c.

65. Let it be required next to solve in entire numbers the equation $49x - 35y = 11$.

Here it will be observed, that the coefficients of x and y have a common factor 7; dividing therefore both members by 7, we have $7x - 5y = \frac{11}{7}$, an equation which is evidently impossible in entire numbers; the proposed therefore does not admit of entire and positive values for x and y. In general, the proposed equation being reduced to the form of the preceding, *if the coefficients of* x *and* y *have a common factor, which does not enter into the second member, the equation is impossible in entire numbers.*

If there be a factor, common to the coefficients of x and y, which does not enter into the second member, and this factor be not perceived at first, the course of the calculation will make known, sooner or later, the impossibility of solving the question in entire numbers.

Applying the process, explained above, to the equations $49x - 35y = 11$, we obtain finally the equation

$$t = 2t' - 1 - \frac{4}{7},$$

an equation, which is evidently impossible in entire numbers for t and t', from which it is readily inferred, that the proposed will not admit of entire solutions.

If the equation of the proposed question has therefore a factor common to both members, we suppress it; the coefficients of x and y will then be prime to each other, if the question admits of solution in entire and positive numbers. This being the case, the process explained above will always lead to a final equation, in which the coefficient of one of the indeterminates is equal to

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unity. Indeed, it will readily be perceived, that in the course pursued we apply to the coefficients of x and y in the proposed the process of the greatest common divisor; since then these coefficients are by hypothesis prime to each other, we arrive finally at a remainder equal to unity, which will be the coefficient of the last but one of the *indeterminates* introduced in the course of the calculation.

66. In certain cases the preceding process admits of simplifications, which it is important to introduce in practice.

1. Let it be required to solve in entire numbers the equation $5x+3y=49$.

Proceeding as before, we have

$$3y=49-5x$$
$$y=16-x-\frac{2x-1}{3};$$

but the quotient on dividing $5x$ by 3 being nearer $2x$, we put the equation under the form

$$3y=49-6x+x;$$

whence $$y=16-2x+\frac{x+1}{3};$$

from which we obtain $$x=3t-1$$
$$y=18-5t.$$

By means of the simplification, here introduced, the number of indeterminates, employed in the calculation, is one less, than would otherwise be necessary.

2. A person purchases wheat at 16*s*. and barley at 9*s*. a bushel, and pays in all 167*s*. How many bushels of each did he purchase? Ans. 2 of wheat and 15 of barley.

3. To find two numbers such, that if the first be multiplied by 7 and the second by 13, the sum of the products will be 128. Ans. The numbers are 9 and 5, or &c.

4. Let it be proposed next to resolve in entire numbers the equation $13x-57y=101$.

Deducing the value of x, we have

$$x = 4y + 7 + \frac{5y + 10}{13} = 4y + 7 + \frac{5(y + 2)}{13}.$$

In order that x and y in this equation may be entire numbers, $\frac{5(y+2)}{13}$ must be equal to an entire number; but since 5 and 13 are prime to each other, it is necessary in order to this, that $\frac{y+2}{13}$ should be an entire number; putting t for this number, we have

$$x = 4y + 7 + 5t$$
$$13t = y + 2.$$

from which we obtain

$$x = 57t - 1$$
$$y = 13t - 2.$$

Here, every entire and positive value for t will give similar values for x and y; but if we suppose $t = 0$, or to be negative, the values of x and y will be negative. The number of entire and positive solutions of the proposed is therefore *infinite*, and the smallest system of values for x and y is

$$x = 56, \; y = 11.$$

By means of the simplifications, here introduced, one indeterminate only is employed, instead of three, which would otherwise be necessary.

5. To divide 100 into two such parts, that if the first be divided by 5, the remainder will be 2; and if the second be divided by 7 the remainder will be 4.

Ans. The parts are 47 and 53, or 12 and 88.

6. To find two numbers such, that 11 times the first diminished by 7 times the second, may be equal to 53.

Ans. 8 and 5, or &c.

7. A person purchases some horses and oxen; he pays \$30 for each horse, and \$23 for each ox; and he finds, that the oxen cost him \$12 more than the horses. How many horses and oxen did he buy? Ans. 18 horses and 24 oxen, or &c.

8. To find two numbers such, that if 8 be added to 17 times the first, the sum will be equal to 49 times the second.

Ans. 37 and 13, or &c.

9. Let it be proposed next to resolve the equation

$$39x - 56y = 11.$$

Deducing from this equation the value of x, we have

$$x = y + \frac{17y + 11}{39}.$$

Here, in the expression $\frac{17y+11}{39}$, it will be observed that the difference between 17, the coefficient of y, and the divisor 39, contains the other term 11 as a factor; on this account we take the quotient $56y$ divided by 39 in excess, which gives

$$x = 2y - \frac{22y - 11}{39} = 2y - \frac{11(2y-1)}{39};$$

from which we readily obtain

$$x = 56t' - 27$$
$$y = 39t' - 19.$$

10. To find two numbers such, that if the first be multiplied by 11 and the second by 17, the first product is 5 greater than the second. Ans. 19 and 12, or &c.

11. In how many ways can a debt of 546 livres be paid, by paying pieces of 15 livres, and receiving in exchange pieces of 11 livres? Ans. The number of ways is infinite. For the first we have 43 of the one, and 9 of the other.

12. The difference between two numbers is 309, and if the greater be divided by 37 the remainder will be 5, and if the less be divided by 54 the remainder will be 2; what are the numbers? Ans. 1337 and 1028, or &c.

67. From what has been done, it will be perceived, that if the equation proposed be of the form $2x + 3y = 10$, the number of solutions in entire and positive numbers will be *limited;* but if the equation be of the form $2x - 3y = 10$, 10 being either positive or negative, the number of solutions will be *infinite.*

If moreover we compare the formulas for x and y with the equations from which they are derived, the coefficient of the indeterminate in the formula for x is the same, it will be observed, with the coefficient of y in the equation; and the coefficient of the indeterminate in the formula for y is the same with the coefficient of x in the equation, taken with the contrary sign, or the converse, as it respects the signs of the coefficients. Having obtained then a first solution of the question proposed, those which follow will be found by adding successively to the values of x the coefficient of y in the equation, and subtracting successively from the values of y the coefficient of x in the equation, or the converse, the coefficients of x and y being taken with the signs, which they have in the equation.

68. We pass next to the solution of problems and equations with three or more unknown quantities.

1. Let it be proposed to pay 741 livres with 41 pieces of money of three different species, viz. pieces of 24 livres, 19 livres, and 10 livres.

Let x, y and z represent respectively the number of pieces of each kind, we have by the question

$$x + y + z = 41$$
$$24x + 19y + 10z = 741.$$

Eliminating one of the unknown quantities, x for example, we have $$5y + 14z = 243.$$

Deducing from this equation formulas for entire values for z and y, according to the method explained above, we have, omitting the accents,

$$z = 5t - 3$$
$$y = 57 - 14t.$$

Substituting next, in the first of the equations of the proposed the expressions for z and y just obtained, and deducing the value of x, we have $x = 9t - 13$.

If we now put for t, in the above formulas for x, y, and z, any entire values whatever, we shall obtain entire values for x, y, and z, which will satisfy the equations of the proposed. But to obtain the entire and positive values only, as the nature of the question requires, it is evident, 1°. that $9t$ must be greater than 13, or which is the same thing, that t must be greater than $1\frac{4}{9}$; 2°. that $14t$ must be less than 57, or which is the same thing, that t must be less than $4\frac{1}{14}$; t can therefore have only the values of 2, 3, 4.

Putting in the formulas above t equal to 2, 3, and 4 successively, we have

$$
\begin{aligned}
x &= 5, 14, 23 \\
y &= 29, 15, 1 \\
z &= 7, 12, 17.
\end{aligned}
$$

The proposed, therefore, admits of three different solutions and of three only.

2. Thirty persons, men, women, and children, spend 80 crowns in a tavern. The share of a man is 7 crowns, that of a woman 5 crowns, and that of a child is 2 crowns. How many persons were there of each class?

Ans. For a first answer we have, 1 man, 5 women, and 24 children.

3. The sum of three entire numbers is 15, and if the first be multiplied by 2, the second by 3 and the third by 7, the sum of the products will be 65. What are the numbers?

Ans. 4, 5, and 6.

From what has been done, it will be easy to see how we are to proceed, in the case of three equations with four unknown quantities, and so on.

SECTION VIII.—SOLUTION OF QUESTIONS IN A GENERAL MANNER.

69. In the solution of a question in numbers, there are, it must have been perceived, two distinct things which require attention. 1°. To determine by a process of reasoning what operations must be performed upon the numbers given in the question in order to obtain the answer sought; 2°. to perform these operations. In the questions which have been solved thus far, the operations have each been performed as soon as determined. Let us now resume the question, art. 1, and instead of performing the operations, as we proceed, let us retain them by means of the proper signs.

Representing as before the less part by x, the greater will be $x+12$, and we have

$$x+x+12=56$$
$$2x=56-12$$
$$x=\frac{56}{2}-\frac{12}{2}.$$

Here the process of reasoning required in the solution of the proposed has been conducted by itself; the expression, at which we arrive, is not the answer sought, but the result of the reasoning pursued; it shows what operations must be performed in order to obtain the answer, viz. that from one half of 56, the number given to be divided, there must be subtracted one half of 12 the given excess. Performing next the operations thus determined, we have 22 for the less part as before.

Let us next resume the sixth question, art. 6; representing again the least part by x, the mean will be $x+40$, the greatest $x+40+60$, and we shall have

$$x+x+40+x+40+60=230$$
$$3x=230-40-40-60$$
$$3x=230-2\times 40-60$$
$$x=\frac{230-2\times 40-60}{3}.$$

Here also the result shows the operations to be performed according to which to find the least part sought, from 230, the number given to be divided, we subtract twice 40, or twice the excess of the mean part above the least, and also 60, the excess of the greatest part above the mean, and take one third of the remainder.

70. If the reasoning pursued in the solution of the preceding questions be examined with attention, it will be perceived, that it does not depend upon the particular numbers given in these questions. It will be precisely the same for any other numbers. The same operations will therefore be necessary to obtain the parts sought, whatever the given number may be.

By preserving the operations, therefore, we resolve the proposed in a *general manner*, that is, we determine once for all what operations are necessary for all questions, which differ from the proposed only in the particular numbers, which are given.

Let it be proposed next to find a number such that the difference between one-ninth and one-seventh of this number shall be equal to 10.

Putting x for the number, we obtain

$$9x - 7x = 7 \times 9 \times 10.$$

Here it will be observed that x is taken 9 times minus seven times, or $9 - 7$ times; $9 - 7$ will, therefore, be the coefficient of x, and the above equation may be written thus:

$$(9-7)x = 7 \times 9 \times 10$$

whence $$x = \frac{7 \times 9 \times 10}{9-7}.$$

Let the following questions be now resolved in a general manner.

1. A company settling their reckoning at a tavern, pay 8*s.* each; but if there had been 4 persons more, they should only have paid 7*s.* each. How many were there?

2. Divide the number 91 into two such parts, that 6 times the first, diminished by 5 times the second, may be equal to 40.

3. Divide the number 56 into two such parts, that one part being divided by 7 and the other by 3, the quotients may together be equal to 10.

71. In the solution of questions in a general manner, according to the method above explained, we should be liable, through inadvertence, to perform some of the operations as we proceed; thus the result would not show how the answer is to be found by means of the numbers originally given in the question. To avoid this inconvenience and at the same time to render the solution more concise, it is usual to represent the given things in a question by signs, which will stand indifferently for the particular numbers given in the question, or for any other numbers whatever.

It is agreed to represent known quantities, or those which are supposed to be given in a question, by the first letters of the alphabet, as a, b, c.

The given things in the question, art. 1, are the number to be divided, and the excess of the greater part above the less; representing these by a and b respectively, the question may be presented generally, thus; *To divide a number* a *into two such parts that the greater may exceed the less by* b.

To resolve the question, thus stated, we denote still the less part by x; the greater will then be $x+b$, and we have

$$x+x+b=a$$
$$2x+b=a$$
$$2x=a-b$$
$$x=\frac{a}{2}-\frac{b}{2}$$

The translation of a formula into common language is called a *rule*. Thus we have the following rule, by which to obtain the less of the parts required according to the question pro-

posed, viz. *From half the number to be divided, subtract half the given excess, the remainder will be the answer.*

Knowing the less part, we obtain the greater by adding to the less the given excess. We may, however, easily obtain a rule for calculating the greater part without the aid of the less. Indeed since the less part is equal to

$\frac{a}{2}-\frac{b}{2}$, if we add b to this, we have $\frac{a}{2}-\frac{b}{2}+b$ equal to the greater. But this expression may, it is easy to see, be reduced to $\frac{a}{2}+\frac{b}{2}$; whence we have the following rule, by which to find the greater part, viz. *To half the number to be divided, add half the given excess, the result will be the answer.*

To apply these rules, let it be required to divide $1753 between two men in such a manner, that the first may have $325 more than the second.

72. The 6th question, art. 6, may be presented in a general manner, thus; *To divide a number* a *into three such parts, that the excess of the mean above the least may be* b, *and the excess of the greatest above the mean may be* c.

Let $x=$ the least part;
then $x+b=$ the mean,
and $x+b+c=$ the greatest,
therefore $\quad x+x+b+x+b+c=a,$
or transposing and reducing $\quad 3x=a-2b-c,$
whence $$x=\frac{a-2b-c}{3}.$$

Translating the above formula into common language, we have the following rule, by which to find the least part, viz. *From the number to be divided, subtract twice the excess of the mean part above the least, and also the excess of the greatest above the mean and take a third of the remainder.*

To obtain a formula for the mean part, we add b, the excess

of the mean above the least, to the above expression for the least part, which gives for the mean

$$\frac{a-2b-c}{3}+b,$$

or reducing to a common denominator

$$\frac{a-2b-c}{3}+\frac{3b}{3};$$

whence we obtain for the mean part

$$\frac{a+b-c}{3}.$$

In like manner the following formula will readily be obtained for the greatest part, viz.

$$\frac{a+b+2c}{3}.$$

Translating these formulas into common language we obtain rules also for the mean and for the greatest part.

1. To apply these rules let it be required to divide \$973 among three men, so that the second shall have \$69 more than the first, and the third \$43 more than the second.

2. A father, who has three sons, leaves them his property amounting to \$15730. The will specifies, that the second shall have \$2320 more than the third, and the eldest shall have \$3575 more than the second. What is the share of each?

73. The operations necessary for the solution of this last question are, it is easy to see, the same with those for the preceding. It may therefore be solved by the same formulas. In like manner the seventh and eighth questions, art. 6, may be solved by the same formulas. This circumstance is worthy attention, since we are thus enabled to comprehend in one the solution of a multitude of questions, differing from each other not only in the particular numbers, which are given, but also in the language, in which they are expressed.

Let now the following questions be generalized.

1. The sum of \$3753 is to be divided among 4 men, in such a manner, that the second will have \$159 more than the first, the

third \$275 more than the second, and the fourth \$389 more than the third. What is the share of each?

2. Three men share a certain sum in the following manner; the sum of A's and B's shares is \$123, that of A's and C's \$110, and that of B's and C's \$83. What is the whole sum and the share of each?

Let $x=$ the whole sum, a, b, and c the sum of the shares of A and B, A and C, B and C, respectively; then $x-a=$ C's share, &c., and we have

$$x=\frac{a+b+c}{2}.$$

74. The seventh question, art. 15, may be stated generally, thus. *A cistern is supplied by two pipes; the first will fill it in* a *hours, the second in* b *hours. In what time will the cistern be filled if both run together?*

Let $x=$ the time; the capacity of the cistern being supposed equal to unity, we have

$$\frac{x}{a}+\frac{x}{b}=1;$$

whence freeing from denominators

$$ax+bx=ab.$$

Here it will be observed, that x is taken a times and also b times; whence on the whole it is taken $a+b$ times; $a+b$ is then the coefficient of x, and the above equation may be written thus,

$$(a+b)\,x=ab;$$

whence $$x=\frac{ab}{a+b}.$$

Translating this formula, we have the following rule for every case of the proposed question, viz. *Divide the product of the numbers, which denote the times employed by each pipe in filling the cistern, by the sum of these numbers; the quotient will be the time required by both the pipes running together to fill the cistern.*

EXAMPLE. Suppose one pipe will fill the cistern in $5\frac{1}{2}$ hours, and the other in 9 hours; in what time will it be filled if both run together?

75. The 4th question, art. 6, may be thus generalized. *A gentleman meeting four poor persons distributed* a *shillings among them; to the second he gave* b *times, to the third* c *times, and to the fourth* d *times as much as to the first. What did he give to each?*

Let x represent what he gave to the first, we then have

$$x+bx+cx+dx=a,$$

or

$$(1+b+c+d)\,x=a;$$

whence

$$x=\frac{a}{1+b+c+d}.$$

Let next the following questions be generalized.

1. A bankrupt wishing to distribute his remaining property among his creditors, finds, that in order to pay them \$175 apiece, he should want \$30, but if he pays them \$168 apiece he will have \$40 left. How many creditors had he?

2. It is required to divide the number 91 into two such parts, that the greater being divided by their difference the quotient may be 7.

3. Divide the number 138 into two such parts, that 5 times the first part diminished by 4 times the second will be equal to 85.

4. Three men, A, B, and C, engage in trade and gain \$500, of which C is to have twice as much as B, and B \$50 less than 4 times as much as A. How much will each receive?

5. A trader having gained \$3450 by his business, and lost \$2375 by bad debts, found, that $\frac{4}{5}$ of what he had left equalled the capital with which he commenced trade. What was his capital?

6. In a certain school $\frac{1}{4}$ of the pupils learn navigation, $\frac{1}{5}$ learn

geometry, $\frac{1}{6}$ learn algebra, and the rest, 23 in number, learn arithmetic. How many pupils are there in all?

76. The nineteenth question, art. 15, may be presented in a general manner, thus. *A laborer was hired for a certain number* a *of days; for each day that he wrought he was to receive* b *shillings, but for each day that he was idle, he was to forfeit* c *shillings. At the end of the time he received* d *shillings. How many days did he work, and how many was he idle?*

Putting $x =$ the number of days, in which he wrought, and resolving the question, we obtain

$$x = \frac{d + ac}{b + c}.$$

EXAMPLE. A laborer was hired for 75 days; for each day that he wrought he was to receive \$3, but for each day that he was idle, he was to forfeit \$7. At the end of the time he received \$125. To determine by the above formula the number of days in which the laborer wrought.

The two following questions may also be solved by the same formula. Why is this the case?

1. A man agreed to carry 20 earthen vessels to a certain place on this condition; that for every one delivered safe he should receive 11 cents, and for every one he broke, he should forfeit 13 cents; he received 124 cents. How many did he break?

2. A fisherman to encourage his son promises him 9 cents for each throw of the net in which he should take any fish, but the son, on the other hand, is to forfeit 5 cents for each unsuccessful throw. After 37 throws the son receives from the father 235 cents. What was the number of successful throws of the net?

77. Let it be proposed next to make a rule for *Fellowship*, and in order to this, let us take the following example.

Three men, A, B, and C commence trade together, and furnish money in proportion to the numbers m, n and p respectively.

they gain a certain sum a. What is each man's share of the gain?

Let $x =$ A's share;

then $\frac{nx}{m} =$ B's, and $\frac{px}{m} =$ C's share.

By the question, therefore,

$$x + \frac{nx}{m} + \frac{px}{m} = a.$$

Freeing from denominators, we have

$$mx + nx + px = ma,$$

or, which is the same thing

$$(m + n + p)\, x = ma;$$

whence $$x = \frac{ma}{m + n + p} = \text{A's share.}$$

Multiplying next the value of x by n, and dividing by m, we obtain

$$\frac{na}{m + n + p} = \text{B's share.}$$

In like manner, we find

$$\frac{pa}{m + n + p} = \text{C's share.}$$

To find a share of the gain therefore; *Multiply the corresponding proportion of the stock into the whole gain, and divide the product by the sum of the proportions.*

78. Let now the following questions be generalized.

1. Three merchants, A, B, C, enter into partnership. A advances \$750, B \$1300 and C \$825. A leaves his money 9 months, B 13 months, and C 15 months in the business. They gain \$830. What is the share of each?

Since A advances \$750 for 9 months, he advances what is equivalent to \$750 $\times$ 9 for 1 month. In like manner B advances what is equivalent to \$1300 $\times$ 13 for one month, &c.

Let p, p', p'' represent respectively the sums advanced by each, and q, q', q'', the times in which these sums were severally

employed; putting a for the sum gained, and x for A's share of the gain, we have

$$x=\frac{pqa}{pq+p'q'+p''q''}.$$

2. A bankrupt leaves $18000 to be divided among three creditors, in proportion to their claims. Now A's claim is to B's as 2 to 3, and B's claim to C's as 4 to 5. How much does each creditor receive?

3. A gentleman hired three men to perform a certain piece of work; the first working 9 hours a day would perform the work in 10 days, the second working 7 hours a day, in 15, and the third, working 12 hours a day, in 6 days. How long will it take them together to perform the work?

4. A merchant purchased 24 yards of cloth of two different kinds for $408. The first cost $18, the second $15 a yard. How many yards were there of each kind?

5. A gentleman hired two workmen for 50 days; to the first he gave $3, to the second $2 a day. On settling with them he paid both together $130. How many days did each work?

What have these last two questions in common, and what general statement will comprehend both?

79. Thus far we have employed the first letters of the alphabet to represent known quantities, and the last to denote those which are unknown. In some cases it is more convenient to represent the quantities, whether known or unknown, by the initials of the words for which they stand.

Let it be proposed to determine what sum of money must be put at interest, at a given rate, in order to amount to a given sum in a given time.

Let $p=$ the principal, or sum put at interest,
$r=$ the rate,
$a=$ the given amount,
$t=$ the given time.

By the question, we have $p + trp = a$,

or $$(1 + tr)p = a;$$

whence $$p = \frac{a}{1+tr}.$$

We have therefore the following rule, by which to find the principal required, viz. *Multiply the rate by the time and add* 1 *to the product; the amount divided by the sum thus obtained will give the principal.*

EXAMPLES. 1. What sum of money must be put at interest at 6 per cent., in order that the principal and interest may, at the end of 5 years, amount to $748,80?

Six per cent. will be $6 on $100, or, $.06 on one dollar; r in the formula will be then, for this case, .06 and we obtain $576 for the answer.

2. A man lent a certain sum of money at 5 per cent.; at the end of 7 years he received for principal and interest $1237.47. What was the sum lent? Ans. $916.65.

3. A merchant finds that by a fortunate speculation with his floating capital, he has gained 15 per cent., and that by this means it has increased to $15571. What was his capital?

Ans. $13540.

80. The equation $p + trp = a$, contains, it will be perceived, four different things, any one of which may be determined, when the others are known. Deducing, for example, the value of t, we have

$$t = \frac{a-p}{rp}.$$

Whence to find the time, when the amount, principal and rate are given; *From the amount subtract the principal, and divide the remainder by the product of the rate multiplied by the principal.*

EXAMPLES. 1. A man put at interest $345 at 4 per cent; at the end of a certain time he received for principal and interest

\$483. Required the time for which the money was lent.

Ans. 10 years.

2. A merchant lets out his floating capital, amounting to \$5873 at 10 per cent. interest. At the end of a certain number of years he finds that he has received \$3523,80 interest. For how many years was his capital let out? Ans. 6.

3. Let the learner prepare the formula and solve the following example.

A gentleman put at interest \$6840, and at the end of 5 years received for capital and interest \$8208. What rate per cent. did he receive? Ans. 4 per cent.

81. In the preceding questions the object has been to determine certain unknown numbers by means of others, which are known, and which have relations to the unknown numbers established by the enunciation of the question. We shall now show the aid derived from the same signs in demonstrating certain properties in relation to known and given numbers.

1. To demonstrate that if both terms of a fraction be multiplied by the same number, the value of the fraction will not be changed.

Let the proposed fraction be designated by $\frac{a}{b}$, and let n be any number whatever.

Putting $\frac{a}{b}=m$, we have $a=bm$;

multiplying both sides of this last by n, we have

$$na=nbm,$$

from which we deduce

$$\frac{na}{nb}=m;$$

whence $$\frac{na}{nb}=\frac{a}{b}.$$

2. Let the same number be added to both terms of a proper

fraction; to determine what effect this will produce upon the value of the fraction.

Let us designate the fraction by $\frac{a}{b}$. Let m be the number added to both terms of this fraction; it then becomes

$$\frac{a+m}{b+m}.$$

To compare the two fractions, it is necessary to reduce them to the same denominator. Performing this operation, we have for the first

$$\frac{ab+am}{bb+bm},$$

and for the second

$$\frac{ab+bm}{bb+bm}.$$

Here the two numerators have the part ab common to both; but the part bm of the second is greater than the part am of the first, since b is greater than a; the second fraction is therefore greater than the first; whence, *If the same number be added to both terms of a proper fraction, the value of the fraction will be increased.*

3. It has been shown in arithmetic that, *Every divisor common to two numbers must divide the remainder after the division of the greater of these numbers by the less.* Let us now demonstrate this property by the aid of algebraic symbols.

Let D be the divisor common to the two numbers; let A D represent the greater of the two numbers and B D the less; let Q be the entire quotient arising from the division of the greater by the less, and let R be the remainder; we have then

$$AD = BD \times Q + R;$$

dividing both sides by D, we have

$$A = B \times Q + \frac{R}{D}.$$

Here the first member of the equation is an entire number,

the second must, therefore, be equal to an entire number; bu. of this member the term BQ is an entire number; whence $\frac{R}{D}$ must be an entire number, that is, R must be exactly divisible by D. The proposition above is, therefore, demonstrated.

The following propositions may now be demonstrated.

1. If the sum of any two quantities be added to their difference, the sum will be twice the greater.

2. If the difference of any two quantities be taken from their sum, the remainder will be twice the less.

3. The second power of the sum of two quantities contains the second power of the first quantity, plus double the product of the first by the second, plus the second power of the second.

4. The second power of the difference of two quantities is composed of the second power of the first quantity, minus the double product of the first by the second, plus the second power of the second.

5. The product of the sum and difference of two quantities is equal to the difference of their second powers.

The questions, art. 15, will furnish additional exercises for the learner in stating and resolving questions in a general manner.

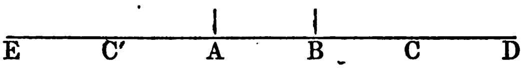
SECTION IX.—Discussion of Problems and Equations of the First Degree.

82. When a problem has been solved in a general manner, it may be proposed to determine what values the unknown quantities will take for particular hypotheses made upon the known quantities. The determination of these different values,

and the interpretation of the results to which we arrive, form what is called the *discussion* of the problem.

The discussion of the following problem presents nearly all the circumstances, that can ever occur in equations of the first degree.

Two couriers set out, at the same time, from two different points A and B, in the line E D, and travel towards D until they meet; the courier, who sets out from the point A, travels at the rate of m miles an hour, the other travels at the rate of n miles an hour; the distance between the points A and B is a miles; at what distance from the points A and B will they meet?



Suppose C to be the point in which they meet; let $x =$ the distance A C, $y =$ the distance B C. We have for the first equation $$x - y = a.$$

The first courier, travelling at the rate of m miles an hour, will be $\frac{x}{m}$ hours in passing over the distance x; the second, travelling at the rate of n miles an hour, will be $\frac{y}{n}$ hours in passing over the distance y; and since these distances must each be passed over in the same time, we shall have for the second equation

$$\frac{x}{m} = \frac{y}{n}.$$

Resolving these two equations, we have

$$x = \frac{am}{m - n}, \quad y = \frac{an}{m - n}.$$

Discussion.

1. Let m be greater than n. In this case the values of x and y will be positive, and the problem will be solved in the exact sense of the enunciation; for, it is evident, that if the courier,

who sets out from A, travels faster than the other, they will meet somewhere in the direction A D.

2. Let n be greater than m. This being the case we shall have

$$x = -\frac{am}{n-m},\ y = -\frac{an}{n-m}.$$

Here the values of x and y are negative. In order to interpret this result, we observe that the courier from B travelling faster than the courier from A, the interval between them must increase continually. It is absurd therefore to require that they should meet in the direction A D. The negative values for x and y indicate, then, an absurdity in the conditions of the question. To show how this absurdity may be done away, let us substitute in the equations of the problem $-x$, $-y$ instead of x and y, we shall then have

$$\left.\begin{array}{r} -x + y = a \\ -\frac{x}{m} = -\frac{y}{n} \end{array}\right\} \text{ or } \left\{\begin{array}{l} y - x = a \\ \frac{x}{m} = \frac{y}{n}. \end{array}\right.$$

The second equation is not affected by the change of sign, as indeed it ought not to be, since it only expresses the equality of the times. In regard to the first, however, we have $y - x = a$, instead of $x - y = a$. This shows that the point, in which the couriers meet, must be nearer to A than to B by the distance A B; it must, therefore, be at some point C′ on the other side of A with respect to B. In order then to remove the absurdity in the enunciation of the question, it is necessary to suppose the couriers, instead of travelling in the direction A D, to travel in the opposite direction B E. Indeed, if we resolve the equations

$$y - x = a$$
$$\frac{x}{m} = \frac{y}{n}$$

we have $x = \frac{am}{n-m}$, $y = \frac{an}{n-m}$, values which are positive, and which answer the conditions of the problem modified, thus,

Two couriers set out at the same time from two points, A and

B, in the line E D, and travel towards E; the courier, who sets out from the point B, travels at the rate of n miles an hour, the other travels at the rate of m miles an hour; the distance between the points B and A is a miles; at what distance from the points B and A will they meet?

3. Let $m=n$. In this case we have $m-n=0$, and the values of x and y become

$$x=\frac{am}{0}, \quad y=\frac{an}{0}.$$

But how shall we interpret this result? Returning to the question, we perceive it to be absolutely impossible to satisfy the enunciation; for, the couriers travelling equally fast, the interval between them must always continue the same, however far they may travel in either direction. It is impossible, then, that they should meet, and no change in the enunciation, so long as we have $m=n$, can make it possible. Indeed, the equations of the problem on the hypothesis $m=n$ become

$$x-y=a$$
$$x-y=0,$$

equations, which are evidently incompatible. Zero being a divisor is, then, a sign of *impossibility*.

The expressions $\frac{am}{0}$, $\frac{an}{0}$ are considered, however, by mathematicians as forming a species of value for x and y, to which they give the name of *infinite* value. To show the reason for this, let us suppose that the difference between m and n without being absolutely nothing is very small; in this case, it is evident that the values of x and y will be very large. Let, for example, $m=3$, $m-n=0.01$, we shall then have $n=2.99$, whence

$$\frac{am}{m-n}=\frac{3a}{.01}=300a, \quad \frac{an}{m-n}=299a.$$

Again let $m-n=.0001$, m being equal to 3, n will then $=2.9999$, whence

$$\frac{am}{m-n}=30000a, \quad \frac{an}{m-n}=29999a.$$

In a word, so long as there is any difference, however small, between m and n, the couriers will meet in one direction or the other; but the distance of the point, in which they meet, from the points A and B will be greater in proportion as the difference between m and n is less. If then *the difference between* m *and* n *is less than any assignable quantity, the distances* $\frac{am}{m-n}$, $\frac{an}{m-n}$ *will be greater than any assignable quantity or* infinite. Since then 0 is less than any assignable quantity, we may employ this character to represent the *ultimate state* of a quantity which may be decreased at pleasure; and since the value of a fractional quantity is greater, in proportion as its denominator is less, the expression $\frac{am}{0}$, and in general, any quantity with zero for a denominator may be considered as the symbol of an *infinite quantity*, that is, a quantity greater than any, which can be assigned.

We say then that the values $x=\frac{am}{0}$, $y=\frac{an}{0}$ are *infinite.*

To show how the notion indicated by the expression $\frac{am}{0}$ does away the absurdity of the equations

$$x-y=a, \; x-y=0,$$

from the second of these equations, we deduce the value of y and substitute it in the first, we then have $x-x=a$; dividing both sides of this last by x, we have

$$1-1=\frac{a}{x}, \text{ or } \frac{a}{x}=0.$$

Here, as we put for x values greater and greater, the fraction $\frac{a}{x}$ will differ less and less from 0, and the equation will approach nearer and nearer to being exact. If then x be greater than any assignable quantity, $\frac{a}{x}$ will be less than any assignable quantity or zero

4. Let us suppose next $m = n$, and at the same time $a = 0$, we shall then have

$$x = \frac{0}{0},\ y = \frac{0}{0}.$$

But how shall we interpret this new result? Returning to the enunciation, we perceive, that if the couriers set out each from the same point and travel equally fast, there is no particular point in which they can be said to meet, since in this case, they will be together through the whole extent of their route. Indeed, on this hypothesis the equations of the problem become

$$x - y = 0,$$
$$x - y = 0,$$

equations which are identical; the problem is therefore *indeterminate*, since we have in fact but one equation with two unknown quantities. The expression $\frac{0}{0}$ is therefore a sign of *indetermination* in the enunciation of the problem.

The preceding hypotheses are the only ones, which lead to remarkable results. They are sufficient to show the manner in which algebra corresponds to all the circumstances in the enunciation of a problem.

GENERAL FORMULAS FOR EQUATIONS OF THE FIRST DEGREE WITH ONE OR TWO UNKNOWN QUANTITIES.

83. Every equation of the first degree with one unknown quantity may, by collecting all the terms which involve x, into one member and the known quantities into the other, be reduced to an equation of the form $Ax = B$, A and B denoting any quantities whatever, positive or negative.

Let there be, for example, the equation

$$\frac{mx}{n} - p = x - q.$$

Freeing from denominators, transposing and uniting terms, we have $(m - n)\,x = n\,(p - q)$.

Comparing this equation with the general formula, we have $m-n=A$, $n(p-q)=B$.

84. Resolving the equation $Ax=B$, we have $x=\frac{B}{A}$. This is a general solution for equations of the first degree with one unknown quantity.

Discussion.

1. Let it be supposed, that in consequence of a particular hypothesis made upon the known quantities, we have $A=0$, the value of x will then be $\frac{B}{0}$. But the equation $Ax=B$ on this hypothesis becomes $0\times x=B$, an equation which, it is evident, cannot be satisfied by any determinate value for x. The equation $0\times x=B$ may, however, be put under the form $\frac{B}{x}=0$. Here, if we consider x greater than any assignable quantity, the fraction $\frac{B}{x}$ will be less than any assignable quantity or zero. On this account we say that *infinity* in this case satisfies the equation. It is evident, at least, that the equation cannot be satisfied by any *finite* value for x.

2. Let us suppose next $A=0$, and at the same time $B=0$, the value of x will then take the form $\frac{0}{0}$. In this case the equation becomes $0\times x=0$, an equation which may be satisfied by any finite quantity whatever, positive or negative. Thus the equation, or the problem, of which it is the algebraic translation, is *indeterminate.*

It should be observed, however, that the symbol $\frac{0}{0}$ does not always indicate that the problem is indeterminate.

Let, for example, the value of x derived from the solution of a problem be

$$x=\frac{a^3-b^3}{a^2-b^2}.$$

If we put $a = b$ in this formula, it will, under its present form, be reduced to $\frac{0}{0}$; but this value for x may be put under the form

$$x = \frac{(a-b)(a^2+ab+b^2)}{(a-b)(a+b)}.$$

If then, before making the hypothesis $a = b$, we suppress the factor $a - b$, the value of x becomes

$$\frac{a^2+ab+b^2}{a+b};$$

from which we obtain $x = \frac{3a}{2}$, on the hypothesis $a = b$.

We conclude therefore that *the symbol* $\frac{0}{0}$ *is sometimes in algebra the sign of the existence of a factor common to the two terms of a fraction, which in consequence of a particular hypothesis becomes* 0, *and reduces the fraction to this form.*

Before deciding then, that the result $\frac{0}{0}$ is a sign that the problem is indeterminate, we must examine whether the expressions for the unknown quantities, which in consequence of a particular hypothesis are reduced to this form, are in their lowest terms, if not, they must be reduced to this state; the particular hypothesis being then made anew, the result $\frac{0}{0}$ shows that the problem is really indeterminate.

85. Every equation of the first degree with two unknown quantities may be reduced to an equation of the form

$$Ax + By = C,$$

A, B, and C denoting any quantities whatever, positive or negative. It is evident, that all equations of the first degree with two unknown quantities may be reduced to this state, 1°. by freeing the equation from denominators; 2°. by collecting into one member all the terms, which involve x and y, and the known quantities into the other; 3°. by uniting the terms,

which contain x into one term, and those which contain y into another.

Let us take the equations

$$Ax + By = C$$
$$A'x + B'y = C'.$$

Resolving these equations we have

$$x = \frac{CB' - BC'}{AB' - BA'}, \quad y = \frac{AC' - CA'}{AB' - BA'}.$$

This is a general solution for all equations of the first degree with two unknown quantities.

To show the use which may be made of these formulas in the solution of equations, let there be the two equations,

$$5x + 3y = 19, \quad 4x + 7y = 29.$$

Comparing these with the general equations, we have

$$A = 5,\ B = 3,\ C = 19,\ A' = 4,\ B' = 7,\ C' = 29,$$

whence, by substitution in the formulas for x and y, we have

$$x = \frac{19 \times 7 - 3 \times 29}{5 \times 7 - 3 \times 4} = \frac{133 - 87}{35 - 12} = \frac{46}{23} = 2$$

$$y = \frac{5 \times 29 - 19 \times 4}{5 \times 7 - 3 \times 4} = \frac{145 - 76}{35 - 12} = \frac{69}{23} = 3.$$

Discussion.

In the above formulas for x and y, let $AB' - BA' = 0$, $CB' - BC'$ and $AC' - CA'$ being each different from zero, we shall then have

$$x = \frac{CB' - BC'}{0}, \quad y = \frac{AC' - CA'}{0}.$$

To interpret these results, we observe that the equation $AB' - BA' = 0$ gives $A' = \frac{AB'}{B}$; substituting this value in the equation $A'x + B'y = C'$, we have

$$\frac{AB'}{B}x + B'y = C';$$

from which we obtain $Ax + By = \frac{BC'}{B'}$; comparing this with the equation $Ax + By = C$, the left hand members, it will be perceived, are identical, while the right are essentially different; for if in the numerator $CB' - BC'$, CB' be greater than BC', C will be greater than $\frac{BC'}{B'}$; and if CB' be less than BC', C will be less than $\frac{BC'}{B'}$. We conclude, therefore, *that the two equations proposed cannot in this case be satisfied, at the same time, by any system whatever of finite values for* x *and* y. The question therefore in this case is *impossible.*

Again, let us suppose $AB' - BA' = 0$, and at the same time, $CB' - BC' = 0$; the value of x in this case is reduced to $\frac{0}{0}$.

To interpret this result, we remark that the equations proposed may, in consequence of the relation $AB - BA' = 0$, be put under the form

$$Ax + By = C$$
$$Ax + By = \frac{BC'}{B'},$$

equations which are identical, since from the relation $CB' - BC' = 0$, we have $\frac{BC'}{B'} = C$.

In order then to resolve the problem, we have in fact but one equation with two unknown quantities; the question therefore is *indeterminate.*

Since the equation $AB' - BA' = 0$ gives $B' = \frac{BA'}{A}$ we have by substitution in the equation $CB' - BC' = 0$.

$$\frac{CBA'}{A} - BC' = 0,$$

or reducing, $AC' - CA' = 0$; *we infer, therefore, that if the value of* x *be of the form* $\frac{0}{0}$, *the value of* y *will be of the same form and the converse.*

PROBLEMS FOR SOLUTION AND DISCUSSION.

1. A merchant has two sorts of wine, one of which costs a, the other b shillings a gallon; from these he would make a mixture of c gallons to be worth d shillings a gallon. How much of each must he take?

Let $x =$ the number of gallons of the first, y of the second,

we have $$x = \frac{c(d-b)}{a-b},\ y = \frac{c(a-d)}{a-b}.$$

How shall we interpret these results 1°. when b or a is equal to d; 2°. when $a = b$; 3°. when $a = b$, and at the same time, $b = d$; 4°. what condition is necessary in order that the question may be solved in the exact sense of the enunciation?

Ans. In the first case, the quantity of one of the ingredients will be 0, as it should be, since, if the price of one of the ingredients is equal to that of the mixture, *none* of the other will be needed to make the mixture of the required price. In the second, since the prices of the ingredients are both the same, a mixture of a different price cannot, it is evident, be made from them; the question, therefore, requires an *impossibility*. In the third case, the price of the ingredients and that of the mixture being each the same, whatever number of gallons be taken of either, the mixture will be of the required price; the question is, therefore, *indeterminate*. The number of solutions is, however, limited by the circumstance that the number of gallons of both ingredients taken together must be equal to c. Finally, in order that the question may be answered in the exact sense of the enunciation, the price of the mixture must be comprised between the prices of the ingredients.

2. To find a number such, that if it be added to the numbers a and b respectively, the first sum will be m times the second.

Putting x for the number, we have $x = \dfrac{mb-a}{1-m}$.

How shall we interpret this result when $m=1$? How when $m=1$, and at the same time $a=b$? How when m is greater than 1, and mb greater than a? What conditions are necessary, in order that the question may be solved in the exact sense of the enunciation?

3. The sum of two numbers is a, and the sum of their products by the numbers m and n respectively is b. What are the numbers?

Putting x and y for the numbers, we have

$$x=\frac{b-na}{m-n},\quad y=\frac{ma-b}{m-n}.$$

How shall we interpret these results, when m is greater than n and na greater than b? How when $m=n$? How when $m=n$, and at the same time $na=b$? What conditions are necessary in order that the question may be solved in the exact sense of the enunciation?

4. Two numbers are in proportion of a to b; but if c be added to both, they will then be in proportion of m to n; what are the numbers?

Ans. $\dfrac{ac(m-n)}{an-bm}$ and $\dfrac{bc(m-n)}{an-bm}$.

SECTION X.—Theory of Inequalities.

86. In the reasonings, which relate to the discussion of a problem, we have frequent occasion to make use of the expressions "*greater than*," "*less than*." In such cases we shall attain a greater degree of conciseness, by representing each of these expressions by a convenient sign. It is agreed to represent the expression "*greater than*" by the sign $>$; thus, *a greater than b* is expressed by $a>b$. The same sign by a change of position

is made to represent the phrase "*less than;*" thus, *x less than b* is expressed by $a < b$.

An equation of the form $a = a$ is called an *eqality.* An expression of the form $a > b$, or $a < b$ is called an *inequality.*

The principles established for equations apply in general to inequalities. As there are some exceptions, however, we shall state the principal transformations, which may be made upon inequalities, together with the exceptions which occur.

1°. *We may always add the same quantity to both members of an inequality, or subtract the same quantity from both members, and the inequality will continue in the same sense.*

Thus, let $3 < 5$; adding 8 to both sides, we have

$$8 + 3 < 5 + 8, \text{ or } 11 < 13.$$

Again let $-3 > -5$; adding 8 to both sides we have

$$8 - 3 > 8 - 5, \text{ or } 5 > 3.$$

This principle enables us, as in the case of equations, to transpose a term from one member of an inequality to the other; thus, from the inequality $a^2 + b^2 > 3c^2 - a^2$, we obtain $2a^2 + b^2 > 3c^2$.

2°. *We may in all cases add, member to member, two or more inequalities established in the same sense, and the inequality, which results, will exist in the sense of the proposed.*

Thus, let there be $a > b$, $c > d$, $e > f$; we have

$$a + c + e > b + d + f.$$

But if we subtract, member from member, two or more inequalities established in the same sense, the inequality, which results, will not always exist in the sense of the proposed.

Let there be the inequalities $4 < 7$, $2 < 3$, we have by subtraction $4 - 2 < 7 - 3$, or $2 < 4$.

But let there be the inequalities $9 < 10$ and $6 < 8$, subtracting the latter from the former, we have

$$9 - 6 > 10 - 8, \text{ or } 3 > 2.$$

3°. *We may multiply or divide the two members of an inequal-*

ity by any positive or absolute number, and the inequality, which results, will exist in the sense of the proposed.

Thus, if we have $a < b$, multiplying both sides by 5, we have $5a < 5b$.

By means of this principle, we may free an inequality from its denominators Thus, let there be

$$\frac{a^2 - b^2}{2d} > \frac{a^2 - b^2}{3a},$$

we have by multiplication $(a^2 - b^2)\,3a > (a^2 - b^2)\,2d$, and by division $3a > 2d$.

But if we multiply or divide the two members of an inequality by a negative quantity, the inequality, which results, will exist in the contrary sense.

Thus, let $8 > 7$; multiplying both sides by -3, we have

$$-24 < -21.$$

From this it follows, that *if we change the sign of each term of an inequality, the inequality, which results, will exist in a sense contrary to that of the proposed; for this transformation will be equivalent to multiplying both members by* -1.

87. Let there now be proposed the inequality

$$7x - \frac{23}{3} > \frac{2}{3}x + 5.$$

Multiplying both sides by 3, we have

$$21x - 23 > 2x + 15;$$

whence transposing and reducing, we have

$$x > 2.$$

Here 2 is the *limit* to the value of x, that is, if we substitute for x in the proposed any value greater than 2, the inequality will be satisfied. The process, by which the limit to the value of the unknown quantity is determined, is called *resolving* the inequality.

EXAMPLES.

1. To find the limit to the value of x in the inequalities

$$14x+\frac{5}{9}>\frac{4x}{5}+230$$

$$x-\frac{25}{7}<10+\frac{2x}{5}.$$

2. To find the limit to the value of x in the inequalities

$$\frac{x}{5}+\frac{x}{3}>\frac{7}{5}+\frac{2x}{9}$$

$$\frac{x}{7}-\frac{x}{14}<\frac{6}{5}-\frac{x}{10}.$$

3. To find the limit to the value of x in the inequalities

$$\frac{ax}{5}+bx-ab>\frac{a^2}{5}$$

$$\frac{bx}{7}-ax+ab<\frac{b^2}{7}.$$

88. The theory of inequalities may be applied to the solution of certain problems.

1. The double of a number diminished by 5 is greater than 25, and triple the number diminished by 7 is less than double the number increased by 13. Required a number that shall possess these properties.

By the question, we have

$$2x-5>25$$
$$3x-7<2x+13.$$

Resolving these inequalities, we have $x>15$, $x<20$. Any number therefore, entire or fractional, comprised between 15 and 20 will satisfy the conditions of the question.

2. A shepherd being asked the number of his sheep replied, that double their number diminished by 7 is greater than 29, and triple their number diminished by 5 is less than double their number increased by 16. Required the number of sheep.

Resolving the question, we have $x > 18$, and $x < 21$. Here all the numbers, comprised between 18 and 21, will satisfy the inequalities; but since the nature of the question requires that the answer should be an entire number, the number of solutions is limited to 2, viz. $x = 19$, $x = 20$.

3. A market woman has a number of oranges, such, that triple the number increased by 2, exceeds double the number increased by 61; and 5 times the number diminished by 70 is less than 4 times the number diminished by 9. How many oranges had she?

4. The sum of two numbers is 32, and if the greater be divided by the less, the quotient will be less than 5 but greater than 2. What are the numbers?

5. The sum of two numbers is 25; if the greater be divided by the less, the quotient will be greater than 3, and if the less be divided by the greater the quotient will be greater than $\frac{1}{5}$. What are the numbers?

SECTION XI.—Extraction of the Square Root.

89. Let it now be proposed to find a number, which multiplied by five times itself, will give a product equal to 125.

Putting x for the number required, we have by the question $5x^2 = 125$, from which we obtain $x^2 = 25$. This equation is essentially different from any, which we have hitherto considered. It is called an equation of the *second degree*, because it contains x raised to the second power. To find the value of x, we must see what number multiplied by itself will give 25. It is obvious, that the number 5 will fulfil this condition; we have therefore $x = 5$.

The value of x is easily found in the present example, but

in others it will be more difficult. Hence arises this new arithmetical question, viz. *To find a number, which multiplied by itself will give a product equal to a proposed number*, or which is the same thing, from the second power of a number to determine the first.

A number, which multiplied by itself will produce a given number, is called the *square* or *second root* of this number. The process for finding the second root is called *extracting* the square or second root.

In the following table, we have the nine primitive numbers with their squares written under them respectively.

1,	2,	3,	4,	5,	6,	7,	8,	9.
1,	4,	9,	16,	25,	36,	49,	64,	81.

By inspection of this table, it will be perceived, that among entire numbers consisting of one or two figures, there are nine only, which are squares of other entire numbers. The remainder have for a root an entire number plus a fraction. Thus 53, which is comprised between 49 and 64, has for its square root 7 plus a fraction.

The numbers in the second line of this table being the squares of those in the first, conversely, the numbers in the first line are the square roots of those in the second. If, therefore, the number, the square root of which is required, consists of one or two figures only, its root will be readily found by means of the table.

Let it be proposed to find the root of a number consisting of more than two figures, 6084, for example.

The square of 9, the largest number consisting of one figure, is 81, and the square of 100, the smallest number consisting of three figures, is 10000; the square root of 6084 will, therefore, consist of two places, viz. units and tens.

To determine then a method, by which to return from the proposed number to its root, let us observe the manner, in which the different parts of a number consisting of two places, 47, for example, are employed in forming the square of this

number. For this purpose we decompose 47 into two parts, viz. 40 and 7, or 4 tens and 7 units. Designating the tens by a and the units by b, we have $a+b=47$, and squaring both sides $a^2+2ab+b^2=2209$. Thus the square of a number, consisting of units and tens, is composed of three parts, viz. *the square of the tens, plus twice the product of the tens multiplied by the units, plus the square of the units.* Thus in 2209, the square of 47, we have

$$\begin{array}{lr} \textit{The square of the tens } (a^2) & =1600 \\ \textit{Twice the tens by the units } (2ab) = & 560 \\ \textit{The square of the units } (b^2) & = \quad 49 \\ \hline & 2209 \end{array}$$

Considering, then, the proposed number 6084 as composed of the square of the tens of the root sought, twice the product of the tens by the units, and the square of the units, if we can discover in this number the first of these parts, viz. the square of the tens, the tens of the root will be readily found. The square of an exact number of tens, it is evident, can have no figure inferior to hundreds. Separating then the two last figures of the proposed from the rest by a comma, the square of the tens will be found in 60, the part at the left of the comma, which, in addition to the hundreds in the square of the tens, will also contain those, which arise from the other parts of the square. 60 is comprised between 49 and 64, the roots of which are 7 and 8 respectively; 7 will, therefore, be the figure denoting the tens in the root sought. Indeed 60 00 is comprised between 49 00, and 64 00, the squares of 70 and 80 respectively; the same is the case with 60 84; the root required will therefore consist of 7 tens and a certain number of units less than ten.

The figure 7 being thus obtained, we place it at the right of the proposed, taking care to separate them by a vertical line; we then subtract 49, the square of 7, from 60, and to the remainder 11 we bring down 84, the two other figures of

the proposed. The result 1184 of this operation will then contain twice the product of the tens of the root by the units, plus the square of the units. Twice the product of the tens by the units will, it is evident, contain no figure inferior to tens. Separating then 4, the right hand figure of the remainder 1184, from the rest by a comma, the part 118 of this remainder, at the left of the comma, must contain the double product of the tens by the units, together with the tens arising from the square of the units.

The double product of the tens is 14; dividing, therefore, 118 by 14, the quotient 8 will be the unit figure exactly, or in consequence of the tens arising from the square of the units, it may be too large by 1 or 2. To determine whether 8 be the right figure for the units of the root, we multiply twice the tens by 8 and subtract the result from 1184, the remainder 64 being equal to the square of 8, shows that 8 is the unit figure sought. We have 78, therefore, for the root required. The operation will stand thus,

$$\begin{array}{r|l}
60,84 & 7 \\
49 & \\
\hline
118,4 & \frac{14}{8} \\
112 & \\
\hline
64 & \\
64 &
\end{array}$$

To complete the root, we place 8, the unit figure, at the right of 7, the figure for the tens. The work, moreover, may be abridged by writing the 8 at the right of the divisor, and multiplying 148 the number thus formed by 8. We thus obtain in one expression twice the tens by the units and the square of the units; this being equal to the remainder 1184 proves, as before, that 8 is the right figure for the units of the root.

With this modification, the work will stand thus,

60,84	78
49	
118,4	148
118 4	

Let us take, as a second example, the number 841. Pursuing the same course as in the preceding example, we find 2 for the tens of the root; subtracting the square of the tens, the remainder will be 441. Separating the unit figure in this remainder from the rest by a comma, and dividing the part at the left by double the tens, in order to obtain the unit figure of the root, we have 11 for the result. This is evidently too much. Indeed, we cannot have more than 9 for the units; we therefore try 9. This proves to be the correct figure. The root sought is therefore 29.

The operation will be as follows:

8,41	29
4	
44,1	49
44 1	

90. Any number however large may be considered as composed of units and tens; 345, for example, may be considered as composed of 34 tens and 5 units.

Let it now be proposed to find the second root of 190969. This number exceeds 10 000 and is less than 1000 000; its root will therefore consist of three places. But from what has been said, the root may be considered as composed of two parts, units and tens. The proposed will, therefore, consist of three parts, viz. the square of the tens of the root, twice the tens by the units and the square of the units. The square of the tens will have no figure inferior to hundreds. Separating, therefore, the last two figures from the rest by a comma, the tens of the root will be found by extracting the square root

of 1909, the part of the proposed at the left of the comma. Regarding 1909 for the moment as a separate number, its root will evidently consist of two places, units and tens. The method of finding the root will, therefore, be the same as in the preceding examples. Performing the necessary operations we obtain 43 for the root and a remainder of 60. There will therefore, be 43 tens in the root of the proposed, and bringing down the last two figures of the proposed by the side of 60, the result 6069 will contain twice the product of the tens of the root sought by the units, plus the square of the units. Separating, therefore, the right hand figure from the rest by a comma, we divide 606, the part on the left of the comma, by 86, twice the tens; this gives 7 for the unit figure. Placing the 7, therefore, at the right of 43, the part of the root already found, and also at the right of 86, and multiplying this last by 7, we have 6069 for the result. 7 is, therefore, the right unit figure, and the root of the proposed is 437.

The following is a table of the operations.

```
19,09,69 | 437
16       |
--------
  309 | 83
  249
--------
  606,9 | 867
  606 9 |
--------
```

The same process, it is easy to see, may be extended to any number however large. From what has been done, therefore, the following rule for the extraction of the second root will be readily inferred, viz. 1°. *Separate the number into parts of two figures each, beginning at the right.* 2°. *Find the greatest second power in the left hand part; write the root as a quotient in division, and subtract the second power from the left hand part.* 3°. *Bring down the two next figures at the right of the remainder for a dividend and double the root already found for a divisor. See how many times the divisor is contained in the*

dividend, neglecting the right hand figure. Write the result in the root at the right of the figure previously found, and also at the right of the divisor. 4°. *Multiply the divisor, thus augmented, by the last figure of the root and subtract the product from the whole dividend.* 5°. *Bring down the next two figures as before, to form a new dividend, and double the root already found for a divisor, and proceed as before. The root will be doubled, if the right hand figure of the last divisor be doubled.*

91. If the number proposed be not a perfect square, we shall obtain by the above rule, the root of the greatest square number contained in the proposed. Thus, let it be required to find the square root of 1287. Applying the rule to this number, we obtain 35 for the root with a remainder 62. This remainder shows that 1287 is not a perfect square. The square of 35 is 1225, that of 36 is 1296; whence 35 is the root of the greatest square contained in the proposed.

92. When the proposed number is not a perfect square a doubt may sometimes arise, whether the root found be that of the greatest square contained in this number. This may be readily determined by the following rule. The square of $a+1$ is a^2+2a+1; whence the square of a quantity greater by unity than a exceeds the square of a by $2a+1$. From this it follows, that *if the root obtained should be augmented by unity or more than unity, the remainder after the operation must be at least equal to twice the root plus unity.* When this is not the case, the root obtained is that of the greatest square contained in the proposed.

EXAMPLES.

1. To find the square root of 56821444. Ans. 7538.
2. To find the square root of 17698849. Ans. 4207.
3. To find the square root of 1607448649. Ans. 40093.
4. To find the square root of 12103441. Ans. 3479.
5. To find the square root of 48303584206084. Ans. 6950078.

93. From what has been done, it will be perceived, that there are many whole numbers, the roots of which are not whole numbers. What is remarkable in regard to these numbers is, that they will have no assignable roots. Thus the numbers 3, 7, 11 have no assignable roots, that is, no number can be found either among whole or fractional numbers, which multiplied by itself will produce either of these numbers. The proof of this depends upon the following proposition, which we shall now demonstrate, viz.

Every number P, *which will exactly divide the product* AB *of two numbers* A *and* B, *and which is prime to one of these numbers must necessarily divide the other number.*

Let us suppose that P will not divide A, and that A is greater than P. Let us apply to A and P the process of the greatest common divisor, designating the quotients, which arise, by $Q, Q', Q'' \dots$ and the remainders by $R, R', R'' \dots$ respectively. It is evident, that if the operation be pursued sufficiently far, we shall obtain a remainder equal to unity, since by hypothesis A and P are prime to each other. This being premised we have the following equations

$$A = PQ + R$$
$$P = RQ' + R'$$
$$R = R'Q'' + R''$$
$$\dots\dots\dots$$
$$\dots\dots\dots$$

Multiplying the first of these equations by B, and dividing by P, we have

$$\frac{AB}{P} = BQ + \frac{BR}{P}.$$

By hypothesis $\frac{AB}{P}$ is an entire number, and since B and Q are each entire numbers the product BQ is an entire number. It follows therefore, that $\frac{BR}{P}$ must be an entire number; whence B multiplied by the remainder R is divisible by P.

Again, multiplying the second of the above equations by B and dividing by P, we have

$$B = \frac{BRQ'}{P} + \frac{BR'}{P}.$$

But we have already shown that $\frac{BR}{P}$ is an entire number, whence $\frac{BRQ'}{P}$ is an entire number. This being the case, $\frac{BR'}{P}$ must be an entire number; whence B multiplied by the remainder R′ must be divisible by P.

If then the remainder R′ is equal to unity, the proposition is demonstrated, since in this case we shall have $B \times 1$ or B divisible by P. But if the remainder R′ is not equal to unity, it is evident, that if the process of the greatest common divisor be applied to the quantities A and P until a remainder is obtained equal to unity, we may in the same manner as above, prove that B multiplied by this remainder will be divisible by P.

We conclude, therefore, that if P, which we have supposed not to divide A, will not divide B, it will not divide AB the product of A by B.

Returning now to our purpose, it is evident, in order that a fractional number $\frac{a}{b}$ may be the root of an entire number c, we must have

$$\frac{a^2}{b^2} = c.$$

But if c be not a perfect square, its root will not be an entire number, that is, a will not be divisible by b; but from what has just been demonstrated, if a is not divisible by b, $a \times a$ or a^2 will not be divisible by b, and by consequence a^2 will not be divisible by b^2; whence $\frac{a^2}{b^2}$ cannot be equal to an entire number c.

94. Though the roots of numbers, which are not perfect squares cannot be assigned either among whole or fractional

numbers, yet, it is evident, there must be a quantity, which multiplied by itself will produce any number whatever. Thus the root of 53 cannot be assigned; yet there must be a quantity, which multiplied by itself will produce 53. This quantity, it is evident, lies between the numbers 7 and 8, for the square of 7 is 49, and the square of 8 is 64. If then we divide the difference between 7 and 8 by means of fractions, we shall obtain numbers, the squares of which will be greater than 49 and less than 64, and which will approach nearer and nearer to 53.

95. All numbers, whether entire or fractional, have a common measure with unity; on this account they are said to be *commensurable;* and since the ratio of these numbers to unity may always be expressed by entire numbers, they are on this account called *rational* numbers.

The root of a number which is not a perfect square can have no common measure with unity; for, since it is impossible to express this root by any fraction, into how many parts soever we conceive unity to be divided, no fraction can be assigned sufficiently small to measure at the same time this root and unity. The roots of numbers, which are not perfect squares, are on this account called *incommensurable* or *irrational* quantities. They are sometimes also called *surds.*

To indicate that the square root of a quantity is to be taken, we use the character $\sqrt{\ }$, which is called a radical sign. Thus $\sqrt{16}$ is equivalent to 4. $\sqrt{2}$ is an *incommensurable* or *surd* quantity.

EXTRACTION OF THE SQUARE ROOT OF FRACTIONS.

96. Since a fraction is raised to the second power by raising the numerator to the second power, and the denominator to the second power, it follows that the square root of a fraction will be found by extracting the square root of the numerator, and of the denominator. Thus, the square root of $\frac{9}{16}$ is $\frac{3}{4}$.

If either the numerator or denominator of the fraction is not a perfect square, the root of the fraction cannot be found exactly. We may, however, always render the denominator of a fraction a perfect square by multiplying both terms of the fraction by the denominator. This will not alter the value of the fraction. The root of the denominator may then be found, and for that of the numerator, we must take the number nearest the root. Thus, if it be required to extract the square root of $\frac{3}{5}$, multiplying both terms by 5, the fraction becomes $\frac{15}{25}$, the root of which is nearest $\frac{4}{5}$, accurate to within less than $\frac{1}{5}$.

If the denominator of the fraction contain a factor, which is a perfect square, it will be sufficient to multiply both terms by the other factor of the denominator. Thus, let it be required to find the square root of $\frac{8}{54}$; multiplying both terms by 6, the fraction becomes $\frac{48}{324}$, the root of which is $\frac{7}{18}$, accurate to within less than $\frac{1}{18}$.

If a greater degree of accuracy is required, we convert the fraction into another, the denominator of which is a perfect square, but greater than that obtained by the method above.

To find, for example, the square root of $\frac{3}{5}$ to within $\frac{1}{15}$, the fraction must be converted into 225ths. This is done by multiplying both terms by 45. Thus we have $\frac{3}{5}=\frac{135}{225}$, the root of which is nearest $\frac{12}{15}$, accurate to within less than $\frac{1}{15}$.

After making the denominator a perfect square, we may multiply both terms of the proposed fraction by any number, which is a perfect square, and thus approximate the root more nearly. If, for example, we multiply both terms of

$\frac{15}{25}$ by 144, the square of 12, we obtain $\frac{2160}{3600}$, the root of which is nearest $\frac{46}{60}$. Thus, we have the root of $\frac{3}{5}$ to within less than $\frac{1}{60}$.

97. We may in this way approximate the roots of whole numbers, the roots of which cannot be exactly assigned.

If it be required, for example, to find the square root of 2; we convert it into a fraction the denominator of which will be a perfect square. Thus, if we put $2 = \frac{450}{225}$, we have for the root $\frac{21}{15}$ or $1\frac{6}{15}$, accurate to within less than $\frac{1}{15}$.

In general, to find the square root of a number accurate to within a given fraction, *we multiply the proposed number by the square of the denominator of the given fraction; we then find the entire part of the square root of this product, and divide the result by the denominator of the given fraction.*

This rule may be demonstrated as follows. Let a be the number proposed, and let it be required to find the root of a to within less than $\frac{1}{n}$.

We shall have, it is evident, $a = \frac{an^2}{n^2}$; let r be the entire part of the root of the numerator of an^2; an^2 will be comprised between r^2 and $(r+1)^2$, and by consequence the square root of a will be comprised between those of $\frac{r^2}{n^2}$ and $\frac{(r+1)^2}{n^2}$, that is to say, between $\frac{r}{n}$ and $\frac{r+1}{n}$; whence $\frac{r}{n}$ will be the root of a to within less than $\frac{1}{n}$.

98. To approximate the root of a number, which is not a perfect square, it will be most convenient to employ some power of 10 as the multiplier of the proposed, or which is the same thing, to convert the proposed into a fraction, the denominator of

which shall be some power of 10. Thus, to approximate the root of 2, let us put $2 = \frac{200}{100}$ or 2.00, the approximate root will be 1.4. Again, let $2 = \frac{20000}{10000}$ or 2.0000, the approximate root will be 1.41.

99. From what has been done, and indeed from the nature of multiplication it follows, that the number of decimal places in the power will be double the decimal places in the root. To find the approximate root of an entire number by the aid of decimals therefore, we must annex to this number twice as many zeros as there are decimal places wanted in the root. Thus, if 5 places are required in the root, ten zeros must be annexed. The zeros may be annexed as we proceed, it being observed, that two zeros must be annexed for every new figure placed in the root.

The root of 7, to three places, will be found as follows.

$$
\begin{array}{l}
7\ (2.645 \\
4 \\
\hline
300 \\
276 \\
\hline
\ \ 2400 \\
\ \ 2096 \\
\hline
\ \ \ \ 30400 \\
\ \ \ \ 26425 \\
\hline
\ \ \ \ \ \ 3975
\end{array}
$$

If the proposed be already a decimal, the number of decimal places must be made even by annexing a zero, if necessary. If the root of the number, thus prepared, is not sufficiently exact, two zeros must be annexed for every new figure required in the root.

100. To find the root of a vulgar fraction by the aid of deci

mals, we convert this fraction into a decimal and then extract the root.

If the proposed consist of an entire part and a fraction, we convert the fraction into a decimal, annex it to the entire part, and then extract the root.

In converting the fraction into a decimal, it will be necessary to pursue the operation, until twice as many decimals are obtained, as are wanted in the root.

EXAMPLES.

1. Find the square root of 11 to within less than $\frac{1}{15}$.

   Ans. $3\frac{4}{15}$.

2. Find the square root of 223 to within less than $\frac{1}{40}$.

   Ans. $14\frac{37}{40}$.

3. Find the square root of 7 to within .01. Ans. 2.64.

4. Find the square root of 227 to within .0001.

   Ans. 15.0665.

5. Find the square root of $\frac{5}{12}$ to 3 places of decimals.

   Ans. 0.645.

6. Finc the approximate square root of $1\frac{3}{4}$. Ans. $1.32+$.

7. Find the approximate square root of $2\frac{13}{15}$.

   Ans. $1.6931+$.

8. Find the approximate square root of 31.027.

   Ans. $5.57+$.

9. Find the approximate square root of 0.01001.

   Ans. $0.10004+$.

10. Find the approximate square root of 3271.4707.

    Ans. $57.19+$

EXTRACTION OF THE SQUARE ROOT OF ALGEBRAIC QUANTITIES.

101. By the rule for multiplication we have

$$(5a^2b^3c)^2 = 5a^2b^3c \times 5a^2b^3c = 25a^4b^6c^2.$$

A monomial is therefore raised to the square by squaring the coefficient and doubling the exponent of each of the letters. Whence to extract the square root of a monomial, it is necessary 1°. *to extract the root of the coefficient;* 2°. *to divide the exponents of each of the letters by* 2.

According to this rule, we have

$$\sqrt{64a^6b^4} = 8a^3b^2.$$

$$\sqrt{625a^2b^8c^6} = 25ab^4c^3.$$

In order that a monomial may be a perfect square, its coefficient, it is evident from the preceding rule, must be a perfect square and the exponent of each of the letters must be an even number.

Thus $98ab^4$ is not a perfect square. Its root can, therefore, be only indicated by means of the radical sign, thus $\sqrt{98ab^4}$. Expressions of this kind are called *irrational* quantities of the second degree, or more simply *radicals* of the second degree.

102. The second power of a product, it is easy to see, is the same as the product of the second powers of all its factors. It follows, therefore, *that the square root of a product will be the same as the product of the square root of all its factors.*

By means of this principle, we may frequently reduce to a more simple form expressions of the kind, which we are here considering. Thus, the above expression $\sqrt{98ab^4}$ may be put under the form $\sqrt{49b^4} \times \sqrt{2a}$; but $\sqrt{49b^4} = 7b^2$, whence $\sqrt{98ab^4} = 7b^2\sqrt{2a}$.

In like manner, we have

$$\sqrt{864a^2b^5c^{11}} = \sqrt{144a^2b^4c^{10} \times 6bc} = 12ab^2c^5\sqrt{6bc}.$$

In the expression $7b^2\sqrt{2a}$, $12ab^2c^5\sqrt{6bc}$, the quantities $7b^2$, $12ab^2c^5$ placed without the radical sign are called the *coefficients* of the radical. The expressions themselves are said to be reduced to their most simple form.

From what has been done, we have the following rule for reducing irrational quantities, consisting of one term, to their most simple form, viz. *Separate the quantity proposed into two parts, one of which shall contain all the factors, which are perfect squares, and the other those which are not; write the roots of the factors, which are perfect squares, without the radical sign as multipliers of the radical quantity, and retain under the radical sign the factors, which are not perfect squares.*

1. To reduce $\sqrt{75a^3bc}$ to its most simple form.

Ans. $5a\sqrt{3abc}$.

2. To reduce $\sqrt{32a^6b^8c}$ to its most simple form.

Ans. $4a^3b^4\sqrt{2c}$.

3. To reduce $\sqrt{175a^6b^3c^7d}$ to its most simple form.

Ans. $5a^3bc^3\sqrt{7bcd}$.

4. To reduce $\sqrt{405a^3b^4c^2de}$ to its most simple form.

Ans. $9ab^2c\sqrt{5ade}$.

5. To reduce $\sqrt{294a^3b^7c^3d^2e^3}$ to its most simple form.

Ans. $7a^4b^3cde\sqrt{6abce}$.

6. To reduce $\sqrt{847a^7b^4c^5d^3}$ to its most simple form.

Ans. $11a^3b^2c^2d\sqrt{7acd}$.

7. To reduce $\sqrt{1014a^5b^9c^3d}$ to its most simple form.

Ans. $13a^2b^4c\sqrt{6abcd}$.

103. The square of $-a$, it will be observed, is a^2, as well as that of $+a$; the root therefore of a^2, may be either $+a$ or $-a$. Both of these roots may be comprehended in one expression by means of the double sign $\pm$. Thus

$$\sqrt{a^2} = \pm a, \quad \sqrt{25b^4c^2} = \pm 5b^2c.$$

The double sign, it is evident, should be considered as affecting the square root of all quantities whatever.

If the monomial proposed be negative, the square root is impossible; since there is no quantity, positive or negative, which multiplied by itself will produce a negative quantity. Thus, $\sqrt{-a}$, $\sqrt{-3b^2}$ are *impossible* or *imaginary* quantities.

Expressions of this kind may be simplified in the same manner as radical expressions, which are real. Thus $\sqrt{-9}$ may be put under the form $\sqrt{-1 \times 9}$; whence

$$\sqrt{-9} = 3\sqrt{-1}.$$

In like manner $\sqrt{-4a^2} = 2a\sqrt{-1}$.

104. We proceed to the extraction of the square root of polynomials.

A quantity consisting of two terms cannot, it is evident, be a perfect square, for the square of a simple quantity will be a simple quantity, and the square of a binomial consists always of three terms.

This being premised, let the proposed be a trinomial, its root, it is evident, will consist of at least two terms. Let $m + n$ be the root, we have $(m+n)^2 = m^2 + 2mn + n^2$.

This shows, that if the proposed be arranged with reference to the powers of some letter that, 1°. the first term of the proposed will be the square of the first term of the root sought; 2°. the second term of the proposed will be equal to twice the first term of the root multiplied by the second; 3°. the third term of the proposed will be the square of the second term of the root.

Let it be proposed to extract the root of the trinomial

$$24a^2b^3c + 16a^4c^2 + 9b^6.$$

Arranging with reference to the letter a, the proposed becomes

$$16a^4c^2 + 24a^2b^3c + 9b^6.$$

In order to obtain the root, we extract according to what has been said the root of the first term $16a^4c^2$, which gives

$4a^2c$. This is the first term of the root. Dividing next the second term $24a^2b^3c$ by $8a^2c$, twice the term of the root already found, we have $3b^3$ for the second term of the root, and since the square of this is equal to $9b^6$ the remaining term of the proposed, the proposed is a perfect square, the root of which is $4a^2c+3b^3$.

Again, let the proposed consist of more than three terms, its root will consist of more than two terms. Let it consist of three and let $m+n+p$ be the root. The expression $m+n+p$ may be put under the form $(m+n)+p$; forming the square after the manner of a binomial, we have for the result $(m+n)^2+2(m+n)p+p^2$, or developing $(m+n)^2$ the result will be $m^2+2mn+n^2+2(m+n)p+p^2$. The proposed, therefore, being arranged with reference to the powers of some letter, it is evident, that the first term of the root will be found by extracting the root of the first term of the proposed, and that the second term of the root will be found by dividing the second term of the proposed by twice the first term of the root already found. If, then, we subtract from the proposed the square of the two terms of the root already obtained, the remainder will be equal to twice the first two terms of the root multiplied by the third plus the square of the third. Dividing this rémainder, therefore, by twice the terms of the root already found, or which is the same thing, dividing the first term of the remainder by twice the first term of the root, we shall obtain the third term sought. Subtracting from the first remainder twice the product of the first two terms of the root by the third, together with the square of the third, if the result be 0, the proposed is a perfect square, and the root is exactly obtained.

Let it be proposed to find the square root of the polynomial $49a^2b^2-24ab^3+25a^4-30a^3b+16b^4$.

The proposed being arranged with reference to the letter a, the work will be as follows:

$$\begin{array}{l|l} 25a^4 - 30a^3b + 49a^2b^2 - 24ab^3 + 16b^4 & 5a^2 - 3ab + 4b^2 \\ 25a^4 - 30a^3b + 9a^2b^2 & \overline{10a^2} \\ \hline \qquad\qquad 40a^2b^2 - 24ab^3 + 16b^4 & \\ \qquad\qquad 40a^2b^2 - 24ab^3 + 16b^4 & \\ \hline \qquad\qquad\qquad 0 & \end{array}$$

We begin by extracting the root of $25a^4$, this gives $5a^2$ for the first term of the root sought, which we place at the right of the proposed and on the same line with it; we then multiply this term of the root by 2 and write the result $10a^2$ under the root. Dividing next the second term of the proposed by $10a^2$ we obtain $-3ab$ for the second term of the root sought. Squaring the part of the root already found, viz. $5a^2-3ab$, and subtracting the square from the proposed, we have for the first term of the remainder $40a^2b^2$. Dividing this last by $10a^2$ the double of $5a^2$, we obtain $4b^2$ for the quotient.

This is the third term of the root sought; forming next the double product of $5a^2-3ab$ by $4b^2$, and subtracting the result together with the square of $4b^2$ from the first remainder the result is 0. The proposed is, therefore, a perfect square, and we have for the root required

$$5a^2 - 3ab + 4b^2.$$

The calculations in the above example may be performed with more facility as follows.

$$\begin{array}{l|l} 25a^4 - 30a^3b + 49a^2b^2 - 24ab^3 + 16b^4 & 5a^2 - 3ab + 4b^2 \\ 25a^4 & \overline{10a^2 - 3ab} \\ \hline \qquad -30a^3b + 49a^2b^2 & 10a^2 - 6ab + 4b^2 \\ \qquad -30a^3b + 9a^2b^2 & \\ \hline \qquad\qquad 40a^2b^2 - 24ab^3 + 16b^4 & \\ \qquad\qquad 40a^2b^2 - 24ab^3 + 16b^4 & \\ \hline \qquad\qquad\qquad 0 & \end{array}$$

Having found the first term $5a^2$ of the rooot, we subtract its square from the first term of the proposed, and bring down the next two terms for a dividend. Dividing the first term of the

dividend by $10a^2$, we obtain $-3ab$, the second term of the root; this we place by the side of $10a^2$; we then multiply the whole, viz. $10a^2-3ab$ by this second term and subtract the result from the dividend, which gives a remainder $40a^2b^2$; to this remainder we bring down the two remaining terms of the proposed for a new dividend. Doubling the two terms of the root already found for a new divisor, we write the result under $10a^2$; dividing next the first term of the new dividend by the first term of the divisor, we obtain $4b^2$ the third term of the root, which we place by the side of the last divisor; we then multiply the whole by this last term of the root, and subtracting the result from the last dividend, 0 remains.

105. The same process, it is easy to see, may be extended to a polynomial of any number of terms whatever.

EXAMPLES.

1. To find the square root of
$4a^4+12a^3x+13a^2x^2+6ax^3+x^4$.
Ans. $2a^2+3ax+x^2$

2. To find the square root of
$9x^4-12x^3+16x^2-8x+4$.
Ans. $3x^2-2x+2$.

3. To find the square root of
$4x^4-16x^3+24x^2-16x+4$.
Ans. $2x^2-4x+2$.

4. To find the square root of
$x^6+4x^5+10x^4+20x^3+25x^2+24x+16$.
Ans. x^3+2x^2+3x+4.

5. To find the square root of
$4x^6+12x^5+5x^4-2x^3+7x^2-2x+1$.
Ans. $2x^3+3x^2-x+1$.

106. The polynomial proposed being arranged with reference to the powers of some letter, if the first term of the

proposed is not a perfect square, or if in the course of the operation we arrive at a remainder, the first term of which is not divisible by twice the first term of the root, the proposed is not a perfect square, and the root cannot be exactly assigned.

The polynomial $a^3b + 4a^2b^2 + 4ab^3$, for example, is not, it is easy to see, a perfect square; the root therefore can only be indicated thus, $\sqrt{a^3b + 4a^2b^2 + 4ab^3}$. We may, however, apply to expressions of this kind the same simplifications, that have already been applied to monomials. The proposed indeed may be put under the form $\sqrt{(a^2 + 4ab + 4b^2)ab}$; but the root of $a^2 + 4ab + 4b^2$ is evidently $a + 2b$, whence

$$\sqrt{a^3b + 4a^2b^2 + 4ab^3} = (a + 2b)\sqrt{ab}.$$

EXAMPLES.

1. To find the square root of $3a^4b - 6a^3b^2 + 3a^2b^3$.
Ans. $a(a - b)\sqrt{3b}$.

2. To find the square root of $5a^2b - 30ab^2 + 45b^3$.
Ans. $(a - 3b)\sqrt{5b}$.

3. To find the square root of $12a^3b^2 + 12a^2b^3 + 3ab^4$.
Ans. $b(2a + b)\sqrt{3a}$.

4. To find the square root of $a^3 + 3a^2b + 3ab^2 + b^3$.
Ans. $(a + b)\sqrt{a + b}$.

5. To find the square root of $a^3 + a^2b - ab^2 - b^3$.
Ans. $(a + b)\sqrt{a - b}$.

SECTION XII.—EQUATIONS OF THE SECOND DEGREE.

107. An equation is said to be of the second degree, when it contains the second power of the unknown quantity, without any of the higher powers.

In an equation of the second degree there can be, therefore, three kinds of terms only, viz. 1°. terms, which involve the second power of the unknown quantity, 2°. terms, which involve the first power of the unknown quantity, 3°. terms consisting entirely of known quantities.

An equation, which contains all three of these different kinds of terms is called a *complete* equation of the second degree.

If the second of these different kinds of terms be wanting, the equation is then called an *incomplete* equation of the second degree.

A complete equation of the second degree is sometimes called an *affected* equation, and an incomplete equation is sometimes called a *pure* equation of the second degree.

108. We are now prepared for the solution of incomplete equations of the second degree.

Let there be proposed, for example, the equation

$$3x^2 - 29 = \frac{x^2}{4} + 510.$$

Freeing from denominators, we have

$$12x^2 - 116 = x^2 + 2040;$$

transposing and uniting terms

$$11x^2 = 2156,$$

or

$$x^2 = 196,$$

whence, extracting the root of both members

$$x = 14.$$

Equations of the second degree, it should be observed, admit of two values for the unknown quantity, while those of the first degree admit of but one only. This arises from the circumstance, that the second power of a quantity will be positive, whether the quantity itself be positive or negative.

Thus we have x in the preceding example equal $+14$ or -14, or, uniting both values in one expression, we have

$$x = \pm 14.$$

Let us take, as a second example, the equation

$$\frac{5}{7}x^2 - 8 = 4 - \frac{2}{3}x^2.$$

Freeing from denominators, transposing and reducing, we have

$$x^2 = \frac{252}{29}, \text{ whence } x = \sqrt{\frac{252}{29}}.$$

In this example $\frac{252}{29}$ is not a perfect square; we can therefore obtain only an approximate value for x.

Let us take, as a third example, the equation

$$x^2 + 25 = 9.$$

Deducing the value of x from this equation, we have

$$x = \sqrt{-16}.$$

To find the value of x, we are here required to extract the square root of -16. But this is impossible; for, as there is no quantity positive or negative, which multiplied by itself will produce a negative quantity, -16, it is evident, cannot have a square root either *exact* or *approximate*. -16 may indeed be considered as arising from the multiplication of $+4$ by -4; but $+4$ and -4 are different quantities; their product therefore is not a square.

The result $x = \sqrt{-16}$ shows then, that it is impossible to resolve the equation, from which it is derived. In general, an expression for the square root of a negative quantity is to be regarded as a symbol of *impossibility*.

109. Equations of the kind, which we are here considering, may always be reduced to an equation of the form $ax^2 = b$, a and b denoting any known quantities whatever, positive or negative. It is evident, that they may be reduced to this state, *by collecting into one member the terms, which involve* x^2 *and reducing them to one term, and collecting the known terms into the other member.*

Resolving the equation $ax^2=b$, we have

$$x=\sqrt{\frac{b}{a}}.$$

This is a general solution for incomplete equations of the second degree.

If $\frac{b}{a}$ be a perfect square, the value of x may be obtained exactly, if not, it may be found with such degree of approximation as we please. If $\frac{b}{a}$ be negative, we shall have $\sqrt{-\frac{b}{a}}$ a symbol of impossibility.

From what has been done, we have the following rule for the solution of incomplete equations of the second degree, viz. *Collect into one member all the terms, which involve the square of the unknown quantity, and the known quantities into the other; free the square of the unknown quantity from the quantities, by which it is multiplied or divided; the value of the unknown quantity will then be obtained by extracting the square root of each member.*

QUESTIONS PRODUCING INCOMPLETE EQUATIONS OF THE SECOND DEGREE.

1. What two numbers are those, whose difference is to the greater as 2 to 9, and the difference of whose squares is 128?

Let $9x=$ the greater and $2x=$ the difference, then, &c.

Ans. 18 and 14.

2. It is required to divide the number 14 into two such parts, that the quotient of the greater part divided by the less may be to the quotient of the less divided by the greater as 48 to 27.

Let $x=$ the greater, then $14-x=$ the less, and we have

$$27\frac{x}{14-x}=48\frac{14-x}{x},$$

or $$27x^2=48(14-x)^2;$$

dividing by 3 to make the coefficients perfect squares

$$9x^2=16(14-x)^2;$$

whence $$3x=4(14-x).$$ Ans. 8 and 6.

3. It is required to divide the number 18 into two such parts, that the squares of these parts may be in the proportion of 25 to 16. Ans. 10 and 8.

4. In a court there are two square grass plots; a side of one of which is 10 yards longer than the side of the other; and their areas are as 25 to 9. What are the lengths of the sides?

Ans. 25 and 15 yards.

5. A person bought two pieces of linen, which together measured 36 yards. Each of them cost as many shillings a yard as there were yards in the piece; and their whole prices were in the proportion of 4 to 1. What were the lengths of the pieces?

Ans. 24 and 12 yards.

6. There is a rectangular field, whose length is to the breadth in the proportion of 6 to 5. A part of this equal to $\frac{1}{6}$ of the whole being planted, there remain for ploughing 625 square yards. What are the dimensions of the field?

Ans. The sides are 30 and 25 yards.

7. Two workmen, A and B, were engaged to work for a certain number of days at different rates. At the end of the time, A who had played 4 of the days, received 75 shillings, but B who had played 7 of the days, received only 48 shillings. Now had B played 4 days, and A played 7 days, they would have received exactly alike. For how many days were they engaged; how many did each work, and what had each per day?

Ans. 19 days; A worked 15, and B 12 days, and
A received 5s. and B 4s. a day.

8. Two travellers, A and B, set out to meet each other, A leaving the town C at the same time that B left D. They travelled the direct road C D, and on meeting, it appeared that A had travelled 18 miles more than B; and that A could have gone B's journey in $15\frac{3}{4}$ days, but B would have been 28 days in performing A's journey. What was the distance between C and D? Ans. 126 miles.

9. A and B carried 100 eggs between them to market and each received the same sum. If A had carried as many as B

he would have received 18 pence for them, and if B had carried only as many as A, he would have received only 8 pence. How many had each? Ans. A 40, B 60.

10. What two numbers are those, whose sum is to the greater as 11 to 7, the difference of their squares being 132?

Ans. 8 and 14.

11. A merchant sold for $960 a certain number of pieces of silk, for which he paid four-fifths as many dollars a piece as there were pieces. He gained $1000 by the sale, how many pieces did he sell? Ans. The question is impossible.

COMPLETE EQUATIONS OF THE SECOND DEGREE.

110. Let us take next the equation $x^2+8x=209$. This is a complete equation of the second degree. The solution of this equation, it is evident, would present no difficulty, if the left hand member were a perfect square. But this is not the case; for the square of a quantity consisting of one term will consist of one term, and the square of a quantity consisting of two terms will contain three terms. Let us then see if x^2+8x can be made a perfect square; for this purpose, it will be recollected, that the three parts which compose the square of a binomial are 1°. *the square of the first term of the binomial*, 2°. *twice the first term multiplied by the second*, 3°. *the square of the second term*. Thus,

$$(x+a)^2=x^2+2ax+a^2.$$

If, then, we compare x^2+8x with $x^2+2ax+a^2$, it is evident that x^2+8x may be considered the first and second terms in the square of a binomial. The first term of this binomial will evidently be x; then as $8x$ must contain twice the first term by the second, the second will be found by dividing $8x$ by $2x$, which gives 4 for the quotient. x^2+8x is, therefore, the first two terms in the square of the binomial $x+4$. If, then, we add 16, the square of 4, to x^2+8x, the left hand member of the proposed, the result $x^2+8x+16$ will be a perfect square.

But if 16 be added to the left hand member, it must also be added to the right in order to preserve the equality; the proposed will then become

$$x^2+8x+16=225.$$

Extracting the root of each member of this last, we have

$$x+4=\pm 15,$$

whence $$x=11,\ x=-19.$$

Let us take, as a second example, the equation

$$x^2-\frac{2}{3}x=15\tfrac{8}{9}.$$

Comparing $x^2-\frac{2}{3}x$ with the square of the binomial $x-a$, viz. $x^2-2ax+a^2$, it is evident, that $x^2-\frac{2}{3}x$ may be considered the first two terms of the square of a binomial. By the same course of reasoning as in the preceding example, we find this binomial to be $x-\frac{1}{3}$. If, then, the square of $\frac{1}{3}$ be added to both sides, the left hand member will be a perfect square, and we have

$$x^2-\frac{2}{3}x+\frac{1}{9}=16.$$

Extracting the root of each member, we have

$$x-\frac{1}{3}=\pm 4;$$

whence $$x=4\tfrac{1}{3},\ x=-3\tfrac{2}{3}.$$

Let us take, as a third example, the equation $x^2+px=q$.

Comparing the left hand member of this equation with $x^2+2ax+a^2$, it is evident, that it may be considered as the first two terms in the square of the binomial $x+\frac{p}{2}$; whence, if the square of $\frac{p}{2}$ be added to both sides, the left hand member will become a perfect square, and we shall have

$$x^2+px+\frac{p^2}{4}=q+\frac{p^2}{4}.$$

Extracting the root of each member

$$x+\frac{p}{2}=\pm\sqrt{q+\frac{p^2}{4}},$$

whence $x=-\frac{p}{2}+\sqrt{q+\frac{p^2}{4}}$, $x=-\frac{p}{2}-\sqrt{q+\frac{p^2}{4}}$.

Making the left hand member a perfect square is called *completing the square.* This is done, as will readily be inferred from the preceding examples, *by adding to both sides the square of one half the coefficient of* x *in the second term.*

Let us take for a fourth example, the equation

$$7-\frac{3}{5}x=\frac{61-x^2}{4x-2}.$$

Freeing from denominators, we have

$$140x-70-12x^2+6x=305-5x^2.$$

Transposing and uniting terms, we have

$$146x-7x^2=375.$$

Or, changing the signs of each term and dividing by the coefficient of x^2,

$$x^2-\frac{146x}{7}=-\frac{375}{7}.$$

Completing the square, we have

$$x^2-\frac{146x}{7}+\frac{5329}{49}=-\frac{375}{7}+\frac{5329}{49}=\frac{2704}{49}.$$

Whence, extracting the root of each member

$$x-\frac{73}{7}=\pm\frac{52}{7}$$

$$x=17\tfrac{6}{7},\ x=3.$$

111. The rule for completing the square applies only, it is evident, to equations of the form $x^2+px=q$, p and q denoting any quantities whatever, positive or negative.

If not already of the form $x^2+px=q$, equations of the kind, which we are here considering, must always be reduced to this form, before completing the square. Thus, in the preceding example, the given equation was reduced, before com-

pleting the square, to $x^2 - \frac{146}{7}x = -\frac{375}{7}$, an equation of the form required.

It is evident, that all complete equations of the second degree may be reduced to the form $x^2 + px = q$, 1°. by collecting all the terms which involve x into the first member and uniting the terms, which contain x^2, into one term, and those which contain x into another, 2°. by changing the signs of each term, if necessary, in order to render that of x^2 positive, 3°. by dividing all the terms by the multiplier of x^2, if it have a multiplier, and multiplying all the terms by the divisor of x^2, if it have a divisor.

Let the equation $\frac{ax}{4} - bx^2 = \frac{cx}{5} + ae$ be reduced to the form $x^2 + px = q$.

Freeing from denominators, we have

$$5ax - 20bx^2 = 4cx + 20ae$$

By transposition $\quad -20bx^2 + 5ax - 4cx = 20ae$

Changing signs $\quad 20bx^2 - 5ax + 4cx = -20ae$

Uniting terms $\quad 20bx^2 - (5a - 4c)x = -20ae$

Dividing by $20b$ $\quad x^2 - \frac{(5a-4c)}{20b}x = -\frac{ae}{b}$.

Comparing this equation with the general formula, we have

$$p = -\frac{(5a-4c)}{20b}, \quad q = -\frac{ae}{b}.$$

From what has been done, we have the following rule for the solution of complete equations of the second degree, viz. 1°. *The equation being reduced to the form* $x^2 + px = q$, *add to both members the square of half the coefficient of* x *in the second term;* 2°. *extract the square root of both members, taking care to give to the root of the second member the double sign* $\pm$*;* 3°. *deduce the value of* x *from the equation, which arises from the last operation.*

EXAMPLES.

1. Given $\frac{2x^2}{3}+3\frac{1}{2}=\frac{x}{2}+8$, to find the values of x.

Ans. $x=3$, or $-2\frac{1}{4}$.

2. Given $4x-\frac{36-x}{x}=46$, to find the values of x.

Ans. $x=12$, or $-.75$.

3. Given $\frac{40}{x-5}+\frac{27}{x}=13$, to find the values of x.

Ans. $x=9$, or $1\frac{2}{13}$.

4. Given $4x-\frac{14-x}{x+1}=14$, to find the values of x.

Ans. $x=4$, or $-1\frac{3}{4}$.

5. Given $\frac{x}{x+60}=\frac{7}{3x-5}$, to find the values of x.

Ans. $x=14$, or -10.

6. Given $\frac{x+3}{2}+\frac{16-2x}{2x-5}=5\frac{1}{5}$, to find the values of x.

Ans. $x=5$, or 6.9.

7. Given $\left.\begin{array}{l} x+4y=14 \\ \text{and } y^2+4x=2y+11 \end{array}\right\}$ to find the values of x and y.

Ans. $x=-46$, or 2; $y=15$, or 3.

8. Given $\left.\begin{array}{l} 2x+3y=19 \\ \text{and } 5x^2-7y^2=62 \end{array}\right\}$ to find the values of x and y.

Ans. $x=5$, or $-36\frac{5}{17}$; $y=3$, or $30\frac{9}{17}$.

9. Given $\left.\begin{array}{l} \frac{2x+7y}{4x}=2y-\frac{51+2x}{10} \\ \text{and } \frac{4x+3y}{16}=y-2 \end{array}\right\}$ to find the values of x and y.

Ans. $x=5$, or $-2\frac{2}{27}$; $y=4$, or $1\frac{289}{351}$.

112. We pass next to the solution of some questions.

1. To find a number such, that if three times this number be added to twice its square, the sum will be 65.

Putting x for the number sought, we have by the question

$$2x^2+3x=65.$$

Dividing by 2, we have $x^2+\frac{3}{2}x=\frac{65}{2}$.

Completing the square, $x^2+\frac{3}{2}x+\frac{9}{16}=\frac{65}{2}+\frac{9}{16}$.

Extracting the root $\qquad x+\frac{3}{4}=\pm\frac{23}{4}$,

whence $\qquad x=5,\ x=-\frac{13}{2}$.

The first value of x satisfies the question in the sense, in which it is enunciated. In order to interpret the second, it will be observed, that if we put $-x$ instead of x in the equation $2x^2+3x=65$, it becomes $2x^2-3x=65$. Resolving this equation, we obtain $x=\frac{13}{2}$, $x=-5$, values of x, which differ from the preceding only in the signs. The number $\frac{13}{2}$ will, therefore, satisfy the conditions of the question modified thus,

To find a number such, that if three times this number be subtracted from twice its square, the remainder will be 65.

2. A person bought some sheep for £72; and found if he had bought 6 more for the same money, he would have paid £1 less for each. How many did he buy?

Let $x=$ the number, we have

$$\frac{72}{x}-\frac{72}{x+6}=1,$$

from which we obtain $x=18$, or -24. To interpret the negative result, we write $-x$ for x in the equation, which becomes

$$\frac{72}{-x}-\frac{72}{-x+6}=1,$$

or which is the same thing

$$\frac{72}{x-6}-\frac{72}{x}=1,$$

an equation, which corresponds to the following enunciation

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A person bought some sheep for £72, and found if he had bought 6 less for the same money, he would have paid £1 more for each. How many did he buy?

The negative values here modify the proposed questions, in a manner analogous to what takes place, as we have already seen, in equations of the first degree.

3. To find a number such, that if 15 be added to its square the sum will be equal to eight times this number.

Putting x for the number sought, we have by the question

$$x^2+15=8x.$$

Resolving this equation, we have

$$x=5,\ x=3.$$

In this example both values of x are positive, and answer directly the conditions of the question, in the sense in which it is enunciated.

4. To find a number such, that if the square of this number be augmented by 5 times the number and also by 6, the result will be 2.

Putting x for the number sought, we have by the question

$$x^2+5x+6=2.$$

Whence, resolving the equation we have

$$x=-1,\ x=-4.$$

The values of x in this example are both negative; the question, therefore, as is evident from inspection, cannot be solved in the sense, in which it is enunciated.

If instead of x we write $-x$ in the equation of the proposed it becomes $x^2-5x+6=2$, from which we obtain $x=1$, $x=4$. The numbers 1 and 4 will, therefore, satisfy the conditions of the proposed modified thus,

To find a number such, that if five times this number be subtracted from its square, and 6 be added to the remainder, the result will be 2.

5. To divide the number 10 into two such parts, that the product of these parts will be 30.

Putting x for one of the parts, $10-x$ will be the other; we have therefore by the question

$$10x-x^2=30.$$

Resolving this equation, we obtain

$$x=5+\sqrt{-5},\ x=5-\sqrt{-5}.$$

This result indicates, that there is some absurdity in the conditions of the question proposed, since in order to obtain the value of x, we must extract the root of a negative quantity, which is impossible.

In order to see in what this absurdity consists, let us examine into what two parts a given number should be divided, in order that the product of these parts may be the greatest possible.

Let us represent the given number by p, the product of the two parts by q, and the difference of the two parts by d; the greater part will then be $\frac{p}{2}+\frac{d}{2}$, and the less $\frac{p}{2}-\frac{d}{2}$, and we shall have

$$\left(\frac{p}{2}+\frac{d}{2}\right)\left(\frac{p}{2}-\frac{d}{2}\right)=q,$$

or

$$\frac{p^2}{4}-\frac{d^2}{4}=q.$$

Here the value of q, it is evident, will be greater as that of d is less; the value of q will, therefore, be the greatest possible when d is zero, that is, *the product will be the greatest possible, when the difference between the two parts is zero, or in other words, when the two parts are equal.*

The greatest possible product, which can be obtained by dividing 10 into two parts and taking their product will be 25. The absurdity of the question above consists, therefore, in requiring, that the product of the two parts, into which 10 is to be divided, should be greater than 25.

113. The following questions will serve as an exercise for the learner.

1. There is a field in the form of a rectangular parallelogram, whose length exceeds the breadth by 16 yards, and it contains 960 square yards. Required the length and breadth.

Ans. 40 and 24 yards.

2. There are two numbers, whose difference is 9, and their sum multiplied by the greater, produces 266. What are those numbers? Ans. 14 and 5.

3. A regiment of soldiers, consisting of 1066 men, is formed into two squares, one of which has four men more in a side than the other. What number of men are in a side of each of the squares? Ans. 21 and 25.

4. Two partners, A and B, gained £18 by trade. A's money was in trade 12 months, and he received for his principal and gain £26. Also B's money, which was £30, was in trade 16 months. What money did A put into trade?

Ans. £20.

5. The plate of a looking glass is 18 inches by 12, and is to be framed with a frame of equal width, whose area is to be equal to that of the glass. Required the width of the frame.

Ans. 3 inches.

6. A grazier bought as many sheep as cost him £60; out of which he reserved 15, and sold the remainder for £54, gaining 2 shillings a head by them. How many sheep did he buy, and what was the price of each?

Ans. 75 sheep, and the price was 16*s*.

7. A person bought two pieces of cloth of different sorts; whereof the finer cost 4 shillings a yard more than the other; for the finer he paid £18; but the coarser, which exceeded the finer in length by 2 yards, cost only £16. How many yards were there in each piece, and what was the price of a yard of each?

Ans. 18 yards of the finer, and 20 of the coarser, and the prices were £1 and 16*s*. respectively.

8. Three merchants, A, B, and C, made a joint stock, by

which they gained a sum less than that stock by £80; A's share of the gain was £60, and his contribution to the stock was £17 more than B's. Also B and C contributed together £325. How much did each contribute?

Ans. 75, 58, and 267 pounds respectively.

9. Two messengers, A and B, were dispatched at the same time to a place 90 miles distant; the former of whom riding one mile an hour more than the other, arrived at the end of his journey an hour before him. At what rate did each travel per hour? Ans. A 10 miles, B 9.

10. The joint stock of two partners, A and B, was $416. A's money was in trade 9 months and B's six months; on dividing their stock and gain, A received $228, and B $252. What was each man's stock? Ans. A's $192, B's $224.

11. A and B sold 130 ells of silk, of which 40 ells were A's and 90 B's, for $42. Now A sold for a dollar $\frac{1}{3}$ of an ell more than B did. How many ells did each sell for a dollar?

Ans. B sold 3 ells, and A $3\frac{1}{3}$ for a dollar.

12. A square court-yard has a rectangular gravel walk round it. The side of the court wants 2 yards of being 6 times the breadth of the gravel-walk; and the number of square yards in the walk exceeds the number of yards in the periphery of the court by 164. Required the area of the court.

Ans. 256 yards.

SECTION XIII.—Discussion of the General Equation and of Problems of the Second Degree.

114. All complete equations of the second degree may, as we have already seen, be reduced to an equation of the form $x^2+Px=Q$, P and Q denoting any known quantities whatever, positive or negative. Resolving this equation, we have

$$x=-\frac{P}{2}\pm\sqrt{Q+\frac{P^2}{4}}.$$

This is a general solution for equations of the second degree. We shall now examine the circumstances, which result from the different hypotheses, which may be made upon the known quantities P and Q. This is the object *of the discussion of the general equation of the second degree.*

115. Any quantity, which substituted for the unknown quantity in an equation of the second degree will satisfy it, is called a *root* of the equation.

Before proceeding to the proposed discussion, we shall show, that every equation of the second degree admits of two values for the unknown quantity, or in other words of two roots, and of two only. In order to this we take the general equation

$$x^2 + Px = Q; \qquad (1)$$

Completing the square, we have

$$x^2 + Px + \frac{P^2}{4} = Q + \frac{P^2}{4}, \text{ or } \left(x + \frac{P}{2}\right)^2 = Q + \frac{P^2}{4}.$$

Let $Q + \frac{P^2}{4} = M^2$, we shall then have

$$\left(x + \frac{P}{2}\right)^2 = M^2; \text{ or } \left(x + \frac{P}{2}\right)^2 - M^2 = 0.$$

But the first member of this equation being the difference between two squares, it may be put under the form

$$\left(x + \frac{P}{2} + M\right)\left(x + \frac{P}{2} - M\right) = 0. \qquad (2)$$

This last equation is, it is evident, a necessary consequence of equation (1) and the converse. Each of these equations will be satisfied therefore by the values of x, which satisfy the other, and by these only. But since the left hand member of equation (2) is composed of two factors, this member will become zero if either of its factors is equal to zero, and thus the equation will be satisfied.

If we suppose $x + \frac{P}{2} - M = 0$, we shall have $x = -\frac{P}{2} + M$.

If we suppose $x + \frac{P}{2} + M = 0$, we shall have $x = -\frac{P}{2} - M$.

Or substituting for M its value, we have

$$x = -\frac{P}{2} + \sqrt{Q + \frac{P^2}{4}}$$

$$x = -\frac{P}{2} - \sqrt{Q + \frac{P^2}{4}}.$$

Since equation (2) can be satisfied only by putting for x a value which will reduce to zero one or the other of the two factors, of which the left hand member is composed, it follows, *that every equation of the second degree admits of two roots or values for the unknown quantity and of two only.*

It follows also from what has been done, *that every equation of the second degree may be decomposed into two binomial factors of the first degree with respect to* x, *having* x *for a common term, and the two roots taken with their signs changed, for the second terms.*

Resolving the equation $x^2 + 8x - 209 = 0$, for example, we have $x = 11$, $x = -19$. Either of these values will satisfy the equation. We have also

$$(x - 11)(x + 19) = x^2 + 8x - 209 = 0.$$

If we add together the two general values for x, found above, the sum, it is evident, will be $-P$; if we multiply them together the product will be $-Q$; whence 1°. *The algebraic sum of the two roots is equal to the coefficient of the second term taken with the contrary sign.* 2°. *The product of the two roots is equal to the second member of the equation, taken also with a contrary sign.*

116. Let us new proceed to the discussion proposed. Resuming the value of x, obtained from the general equation $x^2 + Px = Q$, we have

$$x = -\frac{P}{2} \pm \sqrt{Q + \frac{P^2}{4}}.$$

In order to find the value of this expression, which contains a radical, that is a quantity the root of which is to be extracted, we must be able to extract the root either exactly or by approxi-

mation; $Q+\frac{P^2}{4}$ the quantity placed under the radical sign must, therefore, be positive. But $\frac{P^2}{4}$ will necessarily be positive, whatever the sign of P may be; the sign of the quantity $Q+\frac{P^2}{4}$ will, therefore, depend principally upon that of Q or the quantity in the equation altogether known.

1. This being premised, let Q in the first place be *positive.* In this case P may be either positive or negative, and the general equation may be written under the two forms

$$x^2+Px=+Q,\ x^2-Px=+Q,$$

or uniting both in one

$$x^2\pm Px=+Q;$$

from which we have

$$x=\mp\frac{P}{2}\pm\sqrt{Q+\frac{P^2}{4}}.$$

Here $Q+\frac{P^2}{4}$ will evidently be positive; the value of x may, therefore, be obtained, either exactly or with such degree of approximation as we please.

With respect to the two values of x, the first, viz.

$$x=\mp\frac{P}{2}+\sqrt{Q+\frac{P^2}{4}}$$

will be *positive,* for the square root of $\frac{P^2}{4}$ alone being $\frac{P}{2}$, the square root of $Q+\frac{P^2}{4}$ will be greater than $\frac{P}{2}$, the value of x will, therefore, have the same sign with the radical and will by consequence be positive. This value will answer directly the conditions of the equation, or the problem of which the equation is the algebraic translation.

The second value of x, viz. $x=\mp\frac{P}{2}-\sqrt{Q+\frac{P^2}{4}}$, being also necessarily of the same sign with the radical, will be essen-

tially *negative.* This value, though it satisfies the equation, will not answer the conditions of the question, from which the equation is derived. It belongs to an analogous question corre sponding to the equation, after $-x$ has been introduced instead of x, that is, to $x^2 \mp Px = Q$. Indeed, from this last equation we deduce $x = \pm\frac{P}{2} \pm \sqrt{Q + \frac{P^2}{4}}$ values which do not differ from the preceding, except in the sign. Thus the same equation connects together two questions, which differ from each other only in the sense of certain conditions.

2. Again, let Q be *negative.* The equation will then be of the form $x^2 \pm Px = -Q$, and we have

$$x = \mp\frac{P}{2} \pm \sqrt{\frac{P^2}{4} - Q}.$$

Here, in order that the root of the quantity placed under the radical sign may be taken, or in other words, that the value of x may be *real*, it is evident that Q must not exceed $\frac{P^2}{4}$.

Since moreover $\sqrt{\frac{P^2}{4} - Q}$ is numerically less than $\frac{P}{2}$ it follows, that the values of x will both be *negative*, if P is positive in the equation, that is, if the equation is of the form $x^2 + Px = -Q$, and that they will both be *positive*, if P is negative in the equation, that is, if the equation is of the form $x^2 - Px = -Q$.

Indeed, it may be shown *a priori*, that always *when* Q *is negative, in the second member and* P *negative in the first, the problem will admit of two direct solutions, provided that* Q *does not exceed* $\frac{P^2}{4}$.

The equation $x^2 - Px = -Q$, may, by changing the signs of all the terms, be put under the form

$$Px - x^2 = Q, \text{ or } x(P - x) = Q.$$

But the equation $x(P - x) = Q$ is evidently the algebraic translation of the following enunciation, viz. *To divide a number*

P *into two parts, the product of which shall be equal to a given number* Q. For if we put x for one of the parts, the other part will be $P - x$, and the product of the two parts, will be $x(P - x)$.

This being premised, the enunciation of the problem admits, it is evident, of two direct solutions; for the equation of the problem will be the same, whether x be put for one or the other of the parts; there is no reason then, why the equation, when resolved, should give one of the parts rather than the other; it should therefore give both at the same time.

Moreover, in order that the problem may be possible, it is necessary, that Q should not exceed $\frac{P^2}{4}$; for the greatest possible product of the parts, into which the number P may be divided being equal only to $\frac{P^2}{4}$ it is absurd to require that their product, which we have represented by Q, should be greater than $\frac{P^2}{4}$. We conclude therefore that, in all cases *when the known quantity is negative in the second member, but numerically greater than the square of half the coefficient of the second term, the question proposed is impossible.*

117. The following examples will serve as an exercise upon the different cases, which we have here been considering. What change must be made in the enunciations of the first four questions respectively, in order that the negative solutions may become positive? How must the fifth question be modified, so that the answers shall become positive? In what does the absurdity in the seventh question consist?

1. A company at a tavern had £8, 15*s*. to pay, but two of them having left before the bill was settled, those who remained had each in consequence 10*s*. more to pay. How many were in the company at first? $x = \ldots ; 17\frac{1}{2}$.

2. A man travelled 105 miles, and then found that if he had not travelled so fast by 2 miles an hour, he should have been

6 hours longer in performing the same journey. How many miles did he go per hour? $x = 7 \quad x = -5.$

3. A regiment of foot was ordered to send 216 men on garrison duty, each company being to furnish an equal number; but before the detachment marched, 3 of the companies were sent on another service, when it was found that each company that remained was obliged to furnish 12 additional men, in order to make up the complement 216. How many companies were there in the regiment, and what number of men was each ordered to send at first? $x = 9 \quad x = -6; \ \frac{216}{x} = 24.$

4. A and B set out from two towns, which were distant 247 miles, and travelled the direct road till they met. A went 9 miles a day; and the number of days at the end of which they met, was greater by 3, than the number of miles, which B went in a day. Where between A and B did they meet?

On substituting $-x$ for x in the equations, which pertain respectively to the preceding questions, it will be easy to translate these equations into enunciations analogous to those of the questions proposed; there are questions however, in which it will be very difficult to do this, and the negative solutions in such cases are to be regarded merely as connected with the first in the same equation of the second degree.

5. A gentleman counting the guineas, which he had in his purse, finds that if 24 be added to their square, and 8 times their number be subtracted from 17, the sum and remainder will be equal. How many guineas had he in his purse? $x = 7 \quad x = 1$

6. A set out from C towards D, and travelled 7 miles a day. After he had gone 32 miles, B set out from D towards C, and went every day one-nineteenth of the whole journey; and after he had travelled as many days as he went miles in one day, he met A. Required the distance of the places C and D?

7. The difference of two numbers is 7, and the square of the greater is equal to 25 times the less. What are the numbers?

EXAMINATION OF PARTICULAR CASES.

1. In the general equation let Q be negative, that is, let the equation be of the form $x^2 + Px = -Q$, P being of any sign whatever; if we suppose $Q = \frac{P^2}{4}$, the radical

$$\sqrt{\frac{P^2}{4} - Q}$$

will be reduced to 0, and the values of x will be equal each to $-\frac{P}{2}$. Thus if Q be negative in the equation and equal to $\frac{P^2}{4}$, the values of x will be equal, and will both be positive if P is negative, or both negative if P is positive.

2. In the general formula

$$x = -\frac{P}{2} \pm \sqrt{\frac{P^2}{4} + Q},$$

let $Q = 0$, the values of x will then be $x = 0$, $x = -P$.

3. In the same formula let $P = 0$, we have then

$$x = \pm \sqrt{Q}.$$

that is to say, the values of x will in this case be equal, but of contrary signs, real if Q is positive, and imaginary if Q is negative.

4. Let $P = 0$, $Q = 0$, the values of x will then be each equal to 0.

5. We have next to examine a remarkable case which frequently occurs in the solution of problems of the second degree. For this purpose, let us take the equation

$$Ax^2 + Bx = C.$$

This equation being resolved, gives

$$x = \frac{-B \pm \sqrt{B^2 + 4AC}}{2A}.$$

Let it now be supposed, that in consequence of a particular hypothesis made upon the given things in the question, we have $A = 0$, the values of x then become

$$x = \frac{0}{0}, \quad x = -\frac{2B}{0}$$

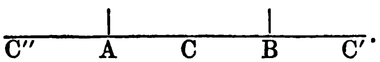
The second value of x here presents itself under the form of infinity, and may be regarded as a true answer, when the question is susceptible of infinite solutions. In order to interpret the first, if we return to the equation, we see that the hypothesis $A = 0$ reduces it to $Bx = C$, from which we deduce $x = \frac{C}{B}$, an expression *finite* and *determinate*, and which must be regarded as the true value of $\frac{0}{0}$ in the present case.

6. Let it be supposed finally, that we have at the same time $A = 0$, $B = 0$, $C = 0$. The equation will then be altogether indeterminate. This is the only case of indetermination, which the equation of the second degree presents.

DISCUSSION OF PROBLEMS.

118. The following problems offer all the circumstances, which usually occur in problems of the second degree.

1. To find on the line A B, which joins two luminous bodies A and B, the point where these bodies shine with equal light.



The solution of this problem depends upon the following principle in physics, viz. The intensity of light from the same luminous body will be, at different distances, in the inverse ratio of the square of the distance.

This being premised, let $a = AB$, the distance between the two bodies; let $b =$ the intensity of A at the unit of distance, $c =$ the intensity of B at the same distance; let C be the point required, and let $AC = x$.

The intensity of A at the distance 1 being b, its intensity at the distance 2, 3, 4, ... will be $\frac{b}{4}, \frac{b}{9}, \frac{b}{16}$ . . . ., and by consequence, at the distance x, it will be $\frac{b}{x^2}$. For the same

reason, the intensity of B at the distance $a-x$ will be $\frac{c}{(a-x)^2}$; whence, by the question, we have

$$\frac{b}{x^2}=\frac{c}{(a-x)^2}.$$

From which, we obtain

$$x=\frac{ab}{b-c}\pm\sqrt{\frac{a^2b^2}{(b-c)^2}-\frac{a^2b}{b-c}}$$

or reducing $$x=\frac{a(b\pm\sqrt{bc})}{b-c}.$$

But $b\pm\sqrt{bc}$ may, it will be observed, be put under the form $\sqrt{b}(\sqrt{b}\pm\sqrt{c})$, and $b-c$ may be put under the form $(\sqrt{b})^2-(\sqrt{c})^2$, or $(\sqrt{b}+\sqrt{c})(\sqrt{b}-\sqrt{c})$.

Taking advantage of this remark, the value of x may be expressed more simply, thus

$$x=\frac{a\sqrt{b}}{\sqrt{b}\pm\sqrt{c}},\text{ whence } a-x=\frac{\pm a\sqrt{c}}{\sqrt{b}\pm\sqrt{c}}$$

or
$$\left.\begin{aligned} x&=\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}\\ x&=\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}\end{aligned}\right\}\text{ and }\left\{\begin{aligned} a-x&=\frac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}}\\ a-x&=\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}.\end{aligned}\right.$$

DISCUSSION.

1. Let b be greater than c.

The first value of x is positive and less than a, since $\frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}$ is a fraction. The point sought, therefore, according to this value of x, is situated between A and B. It is moreover nearer B than A; for, in consequence of $b>c$, we have $\sqrt{b}+\sqrt{b}$, or $2\sqrt{b}>\sqrt{b}+\sqrt{c}$, whence $\frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}>\frac{1}{2}$, and by consequence $\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}>\frac{a}{2}$.

This, indeed, should be the case, since we have supposed the intensity of A greater than that of B.

The corresponding value of $a-x$ is also positive and less as it will be easy to see, than $\frac{a}{2}$.

The second value of x is positive, but greater that a, since we have $\frac{\sqrt{b}}{\sqrt{b}-\sqrt{c}}>1$. This value of x gives, therefore, a second point C′ situated upon AB produced and at the right of A and B. Indeed, since the light from A and B expands itself in all directions, there should be, it is easy to see, on AB produced a second point where A and B shine with equal light. This point moreover should be nearer the body, the light of which is least intense.

The second value of $a-x$ is negative: this should be the case, since we have $x>a$.

2. Let b be less than c.

The first value of x is positive, but less than $\frac{a}{2}$. The corresponding value of $a-x$ is also positive and greater than $\frac{a}{2}$.

Thus on the present hypothesis the point C, situated between A and B, should be nearer A than to B.

The second value of x is essentially negative. In order to interpret it, we return to the equation, which becomes by substituting $-x$ for x, $\frac{b}{x^2}=\frac{c}{(a+x)^2}$. But $a-x$ expressing in the first instance the distance of the point sought from B, $a+x$ must in the present case express the same distance. Thus the point sought should be at the left of A, in C″ for example. Indeed, since by hypothesis the intensity of B is greater than that of A, the second point sought should be nearer A than to B.

3. Let $b=c$.

The first value of x, and also that of $a-x$ is reduced in this case to $\frac{a}{2}$. Thus we have the middle of AB for the point sought. This result conforms to the hypothesis.

The remaining values are reduced to $\frac{a\sqrt{b}}{0}$ or become infinite, that is, the second point where the bodies shine with equal light,

is situated at a distance from A and B greater than any assignable quantity. This result corresponds perfectly with the present hypothesis; for, if we suppose the difference $b - c$, instead of being absolutely nothing, to be very small, the second point will exist, but at a very great distance from A and B. If then $b = c$ or

$$\sqrt{b} - \sqrt{c} = 0$$

the point required must cease to exist, or be placed at an infinite distance.

4. Let $b = c$ and $a = 0$.

The first system of values of x and $a - x$ reduce themselves in this case to 0, and the second system to $\frac{0}{0}$. This last character is here the symbol of indetermination; for, on returning to the equation of the problem,

$$(b - c)\, x^2 - 2abx = - a^2 b,$$

this equation becomes on the present hypothesis

$$0.\, x^2 - 0.\, x = 0,$$

an equation which may be satisfied by any number whatever taken for x. Indeed, since the two bodies have the same intensity and are placed at the same point, they should shine with equal light upon any point whatever in the line AB.

5. Finally let $a = 0$, b being different from c.

Both systems in this case will be reduced to 0, which indicates, that there is but one point, where the bodies shine with equal light, viz. the point, in which the two bodies are situated.

2. To find two numbers such, that the difference of their products by the numbers a and b respectively may be equal to a given number s, and the difference of their squares equal to another given number q.

Denoting by x and y the numbers sought, we have by the question

$$ax - by = s$$
$$x^2 - y^2 = q.$$

Resolving these equations, we have for the first system of values for x and y

$$x=\frac{as+b\sqrt{s^2-q(a^2-b^2)}}{a^2-b^2}$$

$$y=\frac{bs+a\sqrt{s^2-q(a^2-b^2)}}{a^2-b^2}$$

and for the second system, we have

$$x=\frac{as-b\sqrt{s^2-q(a^2-b^2)}}{a^2-b^2}$$

$$y=\frac{bs-a\sqrt{s^2-q(a^2-b^2)}}{a^2-b^2}.$$

DISCUSSION.

1. Let a be greater than b, and by consequence a^2-b^2 positive

In order that the values of x and y may be real, it is necessary that we have

$$q(a^2-b^2)<s^2, \text{ and therefore, } q<\frac{s^2}{a^2-b^2}.$$

This condition being fulfilled, the values of x and y in the first system will be necessarily positive, and will, by consequence, form a *direct solution* of the problem in the sense, in which it is enunciated.

In the second system the value of x will be essentially positive; for $a>b$ gives $as>bs$, and for a still stronger reason, $as>b\sqrt{s^2-q(a^2-b^2)}$.

With respect to the value of y, it may be either positive or negative. In order that it may be positive, we must have

$$bs>a\sqrt{s^2-q(a^2-b^2)}$$

or, squaring both sides,

$$b^2s^2>a^2s^2-a^2q(a^2-b^2)$$

or, adding $a^2 q\,(a^2 - b^2)$ to both sides of this last, and subtracting $b^2 s^2$ from both sides

$$a^2 q\,(a^2 - b^2) > s^2 (a^2 - b^2)$$

or, by division $$q > \frac{s^2}{a^2}.$$

Thus, in order that the second system may be *a real and direct solution*, we must have

$$q < \frac{s^2}{a^2 - b^2}, \text{ but } q > \frac{s^2}{a^2}.$$

If, then, we take for a, b, and s any absolute numbers whatever, provided that we have $a > b$ and that we take for q a number comprised between the two limits $\frac{s^2}{a^2}$ and $\frac{s^2}{a^2 - b^2}$, we shall be certain of obtaining *two direct solutions.*

Thus, let $a = 6$, $b = 4$, $s = 15$; we have

$$\frac{s^2}{a^2} = \frac{225}{36} = 6\tfrac{1}{4}, \text{ and } \frac{s^2}{a^2 - b^2} = \frac{225}{20} = 11\tfrac{1}{4};$$

if then we take $q = 10$, for example, we shall have

$$x = \frac{6 \times 15 \pm 4\sqrt{225 - 20 \times 10}}{20} = \frac{11}{2} \text{ or } \frac{7}{2}$$

$$y = \frac{4 \times 15 \pm 6\sqrt{225 - 20 \times 10}}{20} = \frac{9}{2}, \text{ or } \frac{3}{2}.$$

If on the present hypothesis, we have $q < \frac{s^2}{a^2}$, and for a still stronger reason, $q < \frac{s^2}{a^2 - b^2}$, the value of y in the second system will be negative. This system, therefore, will not be a solution of the proposed problem in the sense, in which it is enunciated, but of an analogous problem, the equations of which are

$$ax + by = s$$
$$x^2 - y^2 = q$$

and which will differ from the proposed in this respect only, that s will express an arithmetical *sum* instead of *difference.*

2. Let a be less than b and therefore $a^2 - b^2$ *negative.*

In this case the expressions for x and y in the first system may be put under the form

$$x=\frac{-as-b\sqrt{s^2+q(b^2-a^2)}}{b^2-a^2}$$

$$y=\frac{-bs-a\sqrt{s^2+q(b^2-a^2)}}{b^2-a^2},$$

and in the second

$$x=\frac{-as+b\sqrt{s^2+q(b^2-a^3)}}{b^2-a^2}$$

$$y=\frac{-bs+a\sqrt{s^2+q(b^2-a^2)}}{b^2-a^2}.$$

The values of x and y in both systems, it is evident, will be real, since the quantity placed under the radical is essentially positive.

In the first system the values of x and y are essentially *negative;* in the second the value of x, it is easy to see, is necessarily positive, but the value of y may be either positive or negative; in order that it may be positive, we must have $q>\frac{s^2}{a^2}$.

3. Let $a=b$, and therefore $a^2-b^2=0$.

On this hypothesis, we have for the first system of values for x and y

$$x=\frac{2as}{0},\ y=\frac{2as}{0};$$

and for the second

$$x=\frac{0}{0},\ y=\frac{0}{0}.$$

Returning to the equations of the proposed in order to interpret these last, we obtain for x and y on the present hypothesis

$$x=\frac{a^2q+s^2}{2as},\ y=\frac{a^2q-s^2}{2as}.$$

PROBLEMS FOR SOLUTION AND DISCUSSION.

1. There are two numbers, whose sum is a and the sum of whose second powers is b. Required the numbers.

Putting x and y for the numbers, we have

$$x=\frac{a\mp\sqrt{2b-a^2}}{2},$$

$$y=\frac{a\pm\sqrt{2b-a^2}}{2}.$$

What conditions are necessary in order that the values of x and y may be real? When will the values of x both be positive? Can both be negative? When will one of them be positive and the other negative, and to what question does the negative value belong?

2. To find two numbers such, that the sum of their products by the numbers a and b respectively may be equal to $2s$, and their product equal to p.

Putting x and y for the numbers, we have

$$x=\frac{s\pm\sqrt{s^2-abp}}{a},$$

$$y=\frac{s\pm\sqrt{s^2-abp}}{b}.$$

What conditions are necessary in order that the values of x and y may be real? What is the greatest value of which p admits? Can either of the values of x or y be negative?

3. To find two numbers such, that the sum of their products by the numbers a and b respectively may be equal to a given number s, and the sum of their squares equal to another given number q.

Putting x and y for the numbers respectively, we have

$$x=\frac{as\pm b\sqrt{(a^2+b^2)q-s^2}}{a^2+b^2},$$

$$y=\frac{bs\pm a\sqrt{(a^2+b^2)q-s^2}}{a^2+b^2}.$$

What conditions are necessary in order that the values of x and y may be real? Within what limits must q be comprised in order that both values of x may be positive? Within what limits must q be comprised in order that both values of y may

be positive? In the second system of values for x and y, when will the value of x be positive, and that of y negative, and what is the analogous problem, to which this system belongs?

4. To find a number such, that its square may be to the product of the differences between this number and two other numbers a and b in the ratio of q to p.

Putting x for the number sought, we have

$$x = \frac{q(a+b) \pm \sqrt{q^2(a-b)^2 + 4pqab}}{2(q-p)}.$$

Let this formula be examined on the different hypotheses $q < p$, $q = p$, $q > p$.

SECTION XIV.—MAXIMA AND MINIMA.

119. In several of the preceding questions, the given things, we have seen, are so connected among themselves, that one is determined by the others to be comprised within certain limits, or to have a greatest or least possible value.

A quantity, the value of which may be made to vary, is called a *variable* quantity; the greatest value of which is called a *maximum* and the least a *minimum*.

Questions frequently occur, in which it is required to determine under what circumstances the result of certain arithmetical operations performed upon numbers will be the greatest or least possible. We shall resolve a few questions of this kind the solutions of which depend upon equations of the second degree.

1. To divide a number, $2a$, into two parts such, that the product of these parts may be a *maximum*.

Let x be one of the parts, then $2a - x$ will be the other, and their product will be $x(2a - x)$. By assigning different values

to x, the product $x(2a-x)$ will vary in magnitude, and the question is to assign to x a value such, that this product may be the greatest possible. Let m be the maximum sought, we have by the question

$$x(2a-x)=m.$$

Regarding for the moment m as known, and deducing from this equation the value of x, we have

$$x=a\pm\sqrt{a^2-m}.$$

From this result it appears, that in order that x may be real, m must not exceed a^2; the greatest value of m will therefore be a^2, in which case we have $x=a$. Thus *to obtain the greatest possible product, the proposed must be divided into two equal parts, and the maximum obtained will be equal to the square of one of these parts.*

In the equation $x(2a-x)=m$, the expressions $x(2a-x)$ is called a *function* of x. This function is itself a *variable*, the value of which depends upon that given to the first variable or x.

2. To divide a number, $2a$, into two parts such, that the sum of the square roots of these parts may be a *maximum*.

Let x^2 be one of the parts, then $2a-x^2$ will be the other, and the sum of the square roots will be $x+\sqrt{2a-x^2}$. Let m be the maximum sought, we have by the question

$$x+\sqrt{2a-x^2}=m;$$

from which we obtain

$$x=\frac{m}{2}\pm\sqrt{\frac{m^2}{4}+\frac{2a-m^2}{2}},$$

or simplifying $$x=\frac{m}{2}\pm\frac{1}{2}\sqrt{4a-m^2}.$$

In order that the values of x may be real, the value of m^2 must not exceed $4a$; $2\sqrt{a}$ is therefore the greatest value, which m can receive.

Let us put $m=2\sqrt{a}$, we have $x=\sqrt{a}$ and $x^2=a$, whence

$2a - x^2 = a$. Thus, *the proposed must be divided into two equal parts in order that the sum of the square roots of the parts may be a maximum.* This maximum moreover will be equal to *twice the square root of one of the parts.*

From what has been done, the following rule for the solution of questions of the kind which we are here considering will readily be inferred, viz. *Having formed the algebraic expression of the quantity susceptible of becoming a* maximum *or* minimum, *make this expression equal to any quantity* m. *If the equation thus made is of the second degree in* x, x *designating the variable quantity, which enters into the algebraic expression, resolve this equation in relation to* x; *make next the quantity under the radical equal to zero, and deduce from this last equation the value of* m; *this will be the* maximum *or* minimum *sought. Substituting finally the value of* m *in the expression for* x, *we obtain the value of* x *proper to satisfy the enunciation proposed.*

If the quantity placed under the radical remains essentially positive, whatever the value of m, we infer that the expression proposed may be of any assignable magnitude whatever, or in other words, that it will have *infinity for a maximum* and *zero for a minimum.*

Thus let there be proposed the expression $\dfrac{4x^2 + 4x - 3}{6(2x + 1)}$; to determine whether this expression is susceptible of a *maximum* or *minimum.*

Putting $\dfrac{4x^2 + 4x - 3}{6(2x + 1)} = m$ and deducing the value of x, we have $x = \dfrac{3m - 1}{2} \pm \dfrac{1}{2}\sqrt{9m^2 + 4}$. Here, whatever value we give to m, the quantity placed under the radical will be positive; the proposed therefore may be of any magnitude whatever.

EXAMPLES FOR PRACTICE.

1. To divide a given number a into two factors, the sum of which shall be a *minimum.*

Ans. The two factors should be equal.

2. Let d be the difference between two numbers; required that the square of the greater divided by the less may be a *minimum*. Ans. The minimum required is $4d$ and the value of the greater part corresponding is $2d$.

3. Let a and b be two numbers of which a is the greater, to find a number such, that if a be added to this number, and b be subtracted from it, the product of the sum and difference thus obtained being divided by the square of the number, the quotient will be a *maximum*.

Ans. The number $= \dfrac{2ab}{a-b}$, and the maximum $= \dfrac{(a+b)^2}{4ab}$.

4. To divide a number $2a$ into two parts such, that the sum of the quotients obtained by dividing the parts mutually, one by the other, may be a *minimum*.

Ans. The number should be divided into two equal parts, and the minimum is 2.

5. To find a number such, that if a and b be added to this number respectively, the product of the two sums thus obtained, divided by the number, may be a *minimum*.

Ans. The number $= \sqrt{ab}$, and the minimum $= (\sqrt{a} + \sqrt{b})^2$.

SECTION XV.—Powers and Roots of Monomials.

120. When a quantity is multiplied into itself, the product, we have seen, is called a *power*, the degree of which is marked by the exponent of the product, thus $aaaaa$ or a^5 is called the fifth power of a; in like manner a^m is called the mth power of a.

The original quantity, from which a power is derived, is called the root of this power. The degree of the root is determined by the number of times the root is found as a factor in the power; thus a is the fifth root of a^5; in like manner a

is the mth root of a^m. The number which marks the degree of the root is called the *index* of the root.

121. Let it be proposed to find the fifth power of $2a^3b^2$; this power is indicated thus, $(2a^3b^2)^5$, and we have, it is evident,

$$(2a^3b^2)^5 = 2a^3b^2 \times 2a^3b^2 \times 2a^3b^2 \times 2a^3b^2 \times 2a^3b^2.$$

Here, it is evident, 1°. that the coefficient 2 must be multiplied into itself four times or raised to the fifth power; 2°. that each one of the exponents of the letters must be added, until it is taken as many times as there are units in the exponent of the power, or in other words, multiplied by 5; we have therefore

$$(2a^3b^2)^5 = 32a^{15}b^{10}.$$

In like manner $\quad (8a^2b^3c)^3 = 512a^6b^9c^3.$

To raise a monomial therefore to any given power, *we raise the coefficient to this power, and multiply each one of the exponents of the letters by the exponent of the power.*

With respect to the sign, with which the powers of a monomial should be affected, it is evident, that whatever be the sign of the quantity itself, its second power will be *positive.* Moreover if the exponent of the power of a monomial be an even number, it is easy to see, that this power may be considered as a power of the square of the proposed quantity. Thus a^8, it is evident, may be considered as the fourth power of a^2; in like manner a^{2m}, any even power of a, may be considered the mth power of a^2. It follows, therefore, *that whatever be the sign of a monomial, any power of it, the exponent of which is an even number, is positive.*

Again, since the power of a simple quantity, the exponent of which is an odd number, is equal to a power of this quantity of an even degree multiplied by the first power, it follows, *that every power of a monomial, the exponent of which is an odd number, will have the same sign as the quantity from which it is formed.*

122. Let it now be proposed to find the third root of $64a^6b^9c^3$.

The root required is indicated thus, $\sqrt[3]{64a^6b^9c^3}$; the sign $\sqrt{\ }$ being employed to denote in general that a root is to be taken, and the index 3 placed above the radical sign to denote the particular root required.

Since the root of a quantity must evidently be sought by a process the reverse of that, by which it is raised to a power, in order to extract the root of a monomial, 1°. *we extract the root of the coefficient*, 2°. *we divide the exponent of each of the letters by the index of the root.*

According to this rule, the third root of the proposed will be $4a^2b^3c$. In like manner the seventh root of $a^{14}b^{21}c^{56}$ is $a^2b^3c^8$.

With respect to the signs, with which the roots of monomials should be affected, it is an evident consequence of the principles already established that,

1°. *Every root of an even degree of a positive monomial may have indifferently either the sign + or —.* Thus, the sixth root of $64a^{18}$ is $\pm 2a^3$.

2°. *Every root, the degree of which is expressed by an odd number, will have the same sign as the quantity proposed.* Thus the fifth root of $-32a^{10}b^5$ is $-2a^2b$.

3°. *Every root of an even degree of a negative monomial is an impossible or imaginary root.* For there is no quantity, which raised to a power of an even degree can give a negative result.

Thus $\sqrt[4]{-a}$, $\sqrt[6]{-b}$ denote impossible or imaginary quantities, in the same manner as $\sqrt{-a}$, $\sqrt{-b}$.

123. From what has been said, it is evident in order that a root may be extracted, 1°. that the coefficient of the proposed must be a perfect power of the degree marked by the index of the root to be extracted; 2°. that the exponents of each of the letters must be divisible by the index of the root.

When this is not the case the root can only be indicated. It should be observed, however, that radical expressions, of any degree whatever admit of the same simplifications as those of the

second degree. These simplifications are founded upon the principle, *that any root whatever of a product is equal to the product of the same root of the several factors.*

Thus let it be proposed to find the third root of $54a^4b^3c^2$ The third root, it is evident, cannot be taken; for 54 is not a perfect third power, and the exponents of the letters a and c are not divisible by 3. We therefore indicate the root, thus, $\sqrt[3]{54a^4b^3c^2}$; but this expression may be put under the form $\sqrt[3]{27a^3b^3 \times 2ac^2}$; whence taking the third root of the factor $27a^3b^3$, we have $\sqrt[3]{54a^4b^3c^2} = 3ab\sqrt[3]{2ac^2}$.

Let it be proposed next to find the third root of

$$125a^5b + 375a^3c.$$

This expression, which it is easy to see is not a perfect third power, may be put under the form $125a^3(a^2b + 3c)$; whence extracting the third root of the first factor, we have for the root sought $5a\sqrt[3]{a^2b + 3c}$.

EXAMPLES.

1. To reduce $\sqrt[4]{80a^5b^7c}$ to its most simple form.
2. To reduce $\sqrt[5]{128x^7y^5z^2}$ to its most simple form.
3. To reduce $\sqrt[3]{ax^3 + bx^6}$ to its most simple form.
4. To reduce $\sqrt[3]{27a^3 + 81a^4b}$ to its most simple form.
5. To reduce $\sqrt[5]{96a^5b^7c^{11}}$ to its most simple form.
6. To reduce $\sqrt[4]{\dfrac{48a^5 - 16a^7b^6}{81a^9 - 324a^4b}}$ to its most simple form.

SECTION XVI.—Powers of Compound Quantities, Theory of Combinations, Binomial Theorem.

124. Powers of compound quantities are found like those of monomials by the continued multiplication of the quantity into itself. They are indicated by inclosing the quantity in a paren-

thesis, to which is annexed the exponent of the power. The third power of $a^2+5ab-bc^2$, for example, is indicated, thus, $(a^2+5ab-bc^2)^3$. This same power may also be indicated thus, $\overline{a^2+5ab-bc^2}|^3$.

Next to monomials, binomials are those, which are the least complicated. We begin therefore with these.

Below are several of the first powers of the binomial $x+a$, viz.

$$
\begin{aligned}
(x+a)^1 &= x+a \\
(x+a)^2 &= x^2+2ax+a^2 \\
(x+a)^3 &= x^3+3ax^2+3a^2x+a^3 \\
(x+a)^4 &= x^4+4ax^3+6a^2x^2+4a^3x+a^4.
\end{aligned}
$$

We have formed the different powers of $x+a$ in this table by the continued multiplication of $x+a$ into itself. In this way we arrive only at particular results. To form any of the higher powers, the process of multiplication must still be continued. This would be tedious, especially, as the power to which the binomial is to be raised, becomes more and more elevated. We proceed, therefore, to investigate a method, by which a binomial may be raised to any power whatever, without the necessity of forming the inferior powers. This method was discovered by Newton. The principle on which it is founded is called the *Binomial Theorem.* The most simple and elementary demonstration of this theorem depends upon the theory of combinations, to which we shall first attend.

THEORY OF COMBINATIONS.

125. The results, obtained by writing one after the other, in every possible way, a given number of letters, in such a manner, that all the letters will enter into each result, are called *permutations.*

Let there be, for example, two letters, a and b. These give, it is evident, two permutations, ab, ba.

Again, let there be three letters, a, b and c. If we set apart one of the letters, a for example, the remaining letters give two permutations, viz. bc, cb; placing next the a at the right of each of these, we have two permutations of three letters, viz. bca, cba; but each of the remaining letters b and c, being set apart in the same manner, will also furnish each two permutations of three lettters; *whence the permutations of three letters will be equal to the permutations of two letters, multiplied by three.*

In like manner *the permutations of four letters will be found equal to the permutations of three letters multiplied by four.*

And in general, *the permutations of any number whatever* n $= 4$ *of letters, will be equal, it is evident, to the permutations of* n — 1 $n-1=3$ *letters, multiplied by* n *the number of letters employed.*

Let Q represent the permutations of $n - 1$ letters, then Qn will represent the permutations of n letters; thus Qn will be a general formula for permutations.

In the general formula Qn, let $n = 2$, then Q will be 1, whence 1×2 will be the permutations of two letters. Again let $n = 3$, then Q will be 1×2; whence $1 \times 2 \times 3$ will be the permutations of three letters. In like manner the permutations of 4 letters will be $1 \times 2 \times 3 \times 4$. The following rule for permutations will, therefore, be readily inferred, viz. *Multiply in order the natural numbers, 1, 2, 3, 4, &c. to the number denoting the letters employed inclusive; the result will be the permutations of the given number of letters.*

126. When a given number of letters are disposed in order one after the other in every possible way, 2 and 2, 3 and 3, and, in general, n and n at a time, the number of letters taken at a time being always less than the given number of letters, the results obtained are called *arrangements.*

Let it be required to form the arrangements of three letters, a, b, and c, two and two at a time.

a,	$a\,b$
	$a\,c$
b,	$b\,a$
	$b\,c$
c,	$c\,a$
	$c\,b$

Setting apart first one of the letters, a for example, we write after this letter each one of the reserved letters b and c; we thus form two of the arrangements sought, viz. ab, ac; setting apart next the letter b, and writing by its side each one of the reserved letters a and c, we form two more of the arrangements sought, viz. ba, bc; pursuing the same course with the remaining letter c, we have in the result, it is plain, all the arrangements required and no more; *whence the arrangements of three letters* 2 *and* 2 *at a time, will be equal to the arrangements of the same letters one at a time, multiplied by the number of letters reserved.*

Let it be required next to form the arrangements of four letters a, b, c, d, three and three at a time.

$a\,b$,	$a\,b\,c$	$c\,a$,	$c\,a\,b$
	$a\,b\,d$		$c\,a\,d$
$a\,c$,	$a\,c\,b$	$c\,b$,	$c\,b\,a$
	$a\,c\,d$		$c\,b\,d$
$a\,d$,	$a\,d\,b$	$c\,d$,	$c\,d\,a$
	$a\,d\,c$		$c\,d\,b$
$b\,a$,	$b\,a\,c$	$d\,a$,	$d\,a\,b$
	$b\,a\,d$		$d\,a\,c$
$b\,c$,	$b\,c\,a$	$d\,b$,	$d\,b\,a$
	$b\,c\,d$		$d\,b\,c$
$b\,d$,	$b\,d\,a$	$d\,c$,	$d\,c\,a$
	$b\,d\,c$		$d\,c\,b$

Having formed the arrangements of the given letters, 2 and

2 at a time, we set apart one of these, ab for example, and write successively by its side each one of the reserved letters c and d, we thus form two of the arrangements sought, viz. abc, abd. The same being done with each one of the remaining arrangements of the given letters, 2 and 2 at a time, we obtain, it is evident, all the arrangements required and no more. Thus, *the arrangements of* 4 *letters taken* 3 *and* 3 *at a time, will be equal to the arrangements of the same letters, taken* 2 *and* 2 *at a time, multiplied by the number of letters reserved.*

In like manner, understanding by letters reserved those which remain, when the given letters are taken one less than the required number at a time, we have *the arrangements of any number* m *of letters, taken* n *and* n *at a time, equal to the arrangements of the same letters,* n — 1 *at a time, multiplied by the number of letters reserved.*

Let P represent the arrangements of m letters $n-1$ at a time; it being required to take the letters n and n at a time, the reserved letters will be $m-(n-1)$, or $m-n+1$; thus the arrangements of m letters, n and n at a time, will be expressed by the formula,

$$P(m-n+1).$$

This will be the general formula for arrangements.

In the general formula $P(m-n+1)$, let n equal 2. In this case P will represent the arrangements of m letters 1 at a time; thus P will equal m; whence $m(m-1)$, will express the arrangements of m letters 2 and 2 at a time.

Again, in the general formula $P(m-n+1)$, let $n=3$. In this case P will represent the arrangements of m letters 2 and 2 at a time; P will therefore equal $m(m-1)$; whence $m(m-1)(m-2)$ will express the arrangements of m letters 3 and 3 at a time.

In like manner the arrangements of m letters 4 and 4 at a time, will be expressed by $m(m-1)(m-2)(m-3)$.

From inspection of the above formulas the following rule for arrangements will be readily inferred, viz. *From the number*

denoting the given letters subtract successively the natural numbers 1, 2, 3, *&c. to the number which denotes the letters to be taken at a time; multiply these several remainders and the number denoting the given letters together; the product will be the arrangements required.*

127. Arrangements, any two of which differ at least by one of the letters, which enter into them, are called *combinations.*

Let it be proposed to determine the number of combinations of three letters, a, b, and c taken two and two at a time.

The arrangements of these letters, two and two at a time, will be

$$\begin{array}{c} ab \\ ba \\ \hline ac \\ ca \\ \hline bc \\ cb \end{array}$$

Among these arrangements we have, it is evident, but three combinations, viz. ab, ac, bc, each one of which is repeated as many times as there are permutations of two letters. Hence *the combinations of three letters taken* 2 *and* 2 *at a time, will be equal to the arrangements of three letters* 2 *and* 2 *at a time, divided by the permutations of two letters.*

In like manner, it will be seen, that the combinations of 4 letters, 3 and 3 at a time, are equal to the arrangements of 4 letters, 3 and 3 at a time, divided by the permutations of three letters.

And in general *the combinations of* m *letters,* n *and* n *at a time, will be equal, it is evident, to the arrangements of* m *letters,* n *and* n *at a time, divided by the permutations of* n *letters.*

From what has been done, we have therefore the following general formula for combinations, viz.

$$\left[\frac{P(m-n+1)}{Qn}.\right]$$

In the general formula let $n=2$, the formula which results will be $\frac{m(m-1)}{1.2}$.

This will give the combinations of m letters 2 and 2 at a time.

Again, let $n=3$; the formula which results will be

$$\frac{m(m-1)(m-2)}{1.2.3}.$$

This will give the combinations of m letters 3 and 3 at a time.

In like manner we obtain $\frac{m(m-1)(m-2)(m-3)}{1.2.3.4}$, a formula which gives the combinations of m letters 4 and 4 at a time.

From inspection of the formulas obtained by making $n=2,3,4$, &c. in the general expression, we may infer a general rule for combinations as has been done already with respect to permutations and arrangements.

1. For how many days can 7 persons be placed in a different position at dinner? 5040.

2. How many words can be made with 5 letters of the alphabet, it being admitted that a number of consonants may make a word? 120.

3. How many combinations can be made of 24 letters of the alphabet taken two and two at a time? 276

4. A general was asked by his king, what reward he should confer on him for his services; the general only desired a farthing for every file of ten men in a file, which he could make with a body of 100 men. At this rate what would he receive?

BINOMIAL THEOREM.

128. If we examine with attention the different powers of $x+a$, art. 124, it will be easy to fix upon the law, according to which the exponents of x and a proceed. But it will not be so easy to determine the law for the numerical coefficients.

If we observe, however, the manner in which the different terms which compose a power are formed, we shall perceive that the numerical coefficients are occasioned by the reduction of several similar terms into one, and that these similar terms arise from the equality of the factors which compose a power. These reductions, it is easy to see, will not take place, if the second terms of the binomials are different. We begin therefore by investigating a law for the formation of the product of any number of binomials $x+a$, $x+b$, $x+c$ . . . , the first terms of which are the same in each, while the second are different.

$$(x+a)\,(x+b)=x^2+\left.\begin{array}{r}a\\b\end{array}\right|x+ab$$

$$(x+a)\,(x+b)\,(x+c)=x^3+\left.\begin{array}{r}a\\b\\c\end{array}\right|x^2+\left.\begin{array}{r}ab\\ac\\bc\end{array}\right|x+abc$$

$$x+a)\,(x+b)\,(x+c)\,(x+d)$$
$$=x^4+\left.\begin{array}{r}a\\b\\c\\d\\ \\ \end{array}\right|x^3+\left.\begin{array}{r}ab\\ac\\ad\\bc\\bd\\ed\end{array}\right|x^2+\left.\begin{array}{r}abc\\abd\\acd\\bcd\\ \\ \end{array}\right|x+abcd.$$

From inspection of the above products, which we have formed by the common rules of multiplication, it will be observed,

1°. *In each product there is one term more than there are units in the number of factors.*

2°. *The exponent of* x *in the first term is the same as the number of factors, and goes on decreasing by unity in each of the following terms.*

3°. *The coefficient of the first term is unity. The coefficient of the second term is equal to the sum of the second terms of the binomials; that of the third term is equal to the sum of the different combinations or products of the second terms of the binomials taken two and two; that of the fourth is equal to the sum of the products of the second terms of the binomials taken three and*

three, and so on. The last term is equal to the product of the second terms of the binomials.

129. We readily infer from analogy, that the same law will obtain, whatever be the number of factors employed. This law may, however, readily be shown to be general. In order to this, it will be sufficient to show, that if the law be true for the product of any number m of binomials, it will also be true for the product of $m + 1$ binomials.

The number of binomial factors being represented by m, the different powers of x will be x^m, x^{m-1}, x^{m-2}, &c. Let A, B, C, . . . U denote the quantities, by which these powers beginning with x^{m-1} are to be multiplied; but as the number of terms must remain indeterminate, until m receives a particular value, we can write only a few of the first and last terms of the expression, designating the intermediate terms by a series of points.

The product of any number m of factors will then be represented by the expression

$$x^m + \mathrm{A}x^{m-1} + \mathrm{B}x^{m-2} + \mathrm{C}x^{m-3} \ldots \mathrm{U}.$$

Multiplying this expression by a new factor $x + \mathrm{K}$, it becomes

$$x^{m+1} \left.\begin{matrix}+\mathrm{A}\\ \mathrm{K}\end{matrix}\right| x^m \left.\begin{matrix}+\mathrm{B}\\ \mathrm{AK}\end{matrix}\right| x^{m-1} \left.\begin{matrix}+\mathrm{C}\\ \mathrm{BK}\end{matrix}\right| x^{m-2} \ldots\ldots \mathrm{UK}.$$

Here the law for the exponents is evidently the same, as in the first expression. With respect to the coefficients, it is evident, 1°. that the coefficient of the first term is unity. 2°. $\mathrm{A} + \mathrm{K}$, the coefficient of the second term, is equal to the sum of the second terms of the $m + 1$ binomials. 3°. Since B by hypothesis expresses the sum of the second terms of the m binomials taken two and two, and AK expresses the sum of the second terms of the m binomials multiplied each by the new second term K, $\mathrm{B} + \mathrm{AK}$, the coefficient of the third term, will be the sum of the products two and two of the second terms of the $m + 1$ binomials.

In the same manner $\mathrm{C} + \mathrm{BK}$, it is easy to see, will be the

sum of the products three and three of the second terms of the $m+1$ binomials, and so on. 4°. The last term UK it is evident, is the product of the $m+1$ second terms.

The law laid down, art. 128, being true therefore for expressions of the fourth degree will, from what has just been demonstrated, be true for those of the fifth; and being true for expressions of the fifth degree, it will be true for those of the sixth and so on; thus it is general.

130. If in the different products which have been formed, art. 128, we make b, c and d each equal to a, these products will be converted into powers of $x+a$, thus

$$(x+a)(x+b)=(x+a)^2=x^2+\begin{array}{r|l} a & x+a^2. \\ a & \end{array}$$

$$(x+a)(x+b)(x+c)=(x+a)^3=x^3+\begin{array}{r|c|l} a & x^2+a^2 & x+a^3. \\ a & a^2 & \\ a & a^2 & \end{array}$$

$$x+a)(x+b)(x+c)(x+d)=(x+a)^4$$
$$=x^4+\begin{array}{r|c|c|l} a & x^3+a^2 & x^2+a^3 & x+a^4. \\ a & a^2 & a^3 & \\ a & a^2 & a^3 & \\ a & a^2 & a^3 & \\ & a^2 & & \\ & a^2 & & \end{array}$$

Comparing these expressions with the different products, from which they are derived, we perceive 1°. that the multiplier of x in the second term has been converted into the first power of a, repeated as many times as there are units in the number of binomials employed, or which is the same thing, as there are units in the exponent of x in the first term. 2°. That the multiplier of the third term has been converted into the second power of a, repeated as many times, as there can be formed different products from a number of letters equal to the number of binomials employed, taken two and two at a time. 3°. That the multiplier of the fourth term has been converted into the third power of a, repeated as many times as

there can be formed different products from a number of letters, equal to the number of binomials employed, taken three and three at a time, and so on.

131. From what has been done, it is evident, therefore, that whatever be the power to which a binomial $x+a$ is to be raised, 1°. the exponent of x in the first term will be equal to the exponent of the power, and that it will go on decreasing by unity in each of the following terms to the last, in which it will be 0. 2°. That the exponent of a in the first term will be 0, in the second unity, and that it will go on increasing by unity, until it becomes equal to the exponent of the power to be formed. 3°. The numerical coefficient of x in the first term will be unity, in the second it will be equal to the exponent of x in the first term, in the third it will be equal to the number of products, which may be formed from a number of letters, equal to the exponent of x in the first term, taken two and two at a time, in the fourth it will be equal to the number of products, which may be formed from the same number of letters, taken three and three at a time and so on.

Let it be required to form the 5th power of $x+a$. The different terms, without the numerical coefficients, will be by the preceding rule $\quad x^5+ax^4+a^2x^3+a^3x^2+a^4x+a^5.$

With respect to the numerical coefficients, that of the first term will be 1, that of the second will be 5, that of the third will be equal to the number of products, which may be formed of 5 letters taken 2 and 2, that of the fourth will be equal to the number of products, which may be formed of 5 letters taken 3 and 3, and so on. Thus the numerical coefficients will be

$$1,\ 5,\ 10,\ 10,\ 5,\ 1;$$

whence

$$(x+a)^5=x^5+5ax^4+10a^2x^3+10a^3x^2+5a^4x+a^5.$$

Let it next be required to raise $x+a$ to the mth power, we shall have, according to the preceding rule, for a few of the first terms without the numerical coefficients

$$x^m+ax^{m-1}+a^2x^{m-2}+a^3x^{m-3}+\ldots,$$

Here the numerical coefficients cannot be determined until we assign a particular value to m; by the preceding rule, however, the numerical coefficient of the second term will be equal to m, whatever the value of m may be. In the development therefore of $(x+a)^m$ we write m for the coefficient of the second term. With respect to the third term the numerical coefficient will be equal to the number of products, which may be formed of m letters 2 and 2 at a time; this is expressed by the formula $\frac{m(m-1)}{1.2}$ we write therefore $\frac{m(m-1)}{1.2}$ for the coefficient of the third term. For a similar reason $\frac{m(m-1)(m-2)}{1.2.3}$ will be the coefficient of the fourth term, and so on. We have then

$$(x+a)^m = x^m + max^{m-1} + \frac{m(m-1)}{1.2}a^2x^{m-2} + \frac{m(m-1)(m-2)}{1.2.3}a^3x^{m-3}$$
$$\ldots\ldots a^m.$$

From inspection of the different terms of this development it will be perceived, that the coefficient of the fourth, for example, is formed by multiplying $\frac{m(m-1)}{1.2}$, the coefficient of the third term, by $m-2$ the exponent of x in this term, and dividing by 3 the number, which marks the place of this term. It will be perceived, also, that the coefficient of the third term is formed in the same manner by means of the second term, and that of the second by means of the first. We readily infer, therefore, the following rule, by which to form the coefficient of any term whatever, viz. *Multiply the coefficient of the preceding term by the exponent of* x *in that term, and divide the product by the number which marks the place of that term from the left.*

From what has been done, we have therefore the following rule, by which to raise a binomial to any power whatever, viz. 1°. *The coefficient of* x *in the first term is unity, and its exponent is equal to the number of units in the degree of the power to which the binomial is to be raised.* 2°. *To pass from any term to the*

following, we multiply the numerical coefficient by the exponent of X *in that term, divide by the number which marks the place of that term from the left, increase by unity the exponent of* A *and diminish by unity the exponent of* X.

According to this rule

$$(x+a)^6 = x^6 + 6ax^5 + 15a^2x^4 + 20a^3x^3 + 15a^4x^2 + 6a^5x + a^6.$$

132. It sometimes happens, that the terms of the proposed binomial are affected with coefficients and exponents. The following example will exhibit the course to be pursued in cases of this kind.

Let it be proposed to raise the binomial $4a^2b - 3abc$ to the fourth power.

Putting $4a^2b = x$, and $-3abc = y$, we have

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

Substituting next for x and y their values, we have

$$(4a^2b - 3abc)^4 = (4a^2b)^4 + 4\,(4a^2b)^3\,(-3abc) + \ldots\ldots$$
$$6\,(4a^2b)^2\,(-3abc)^2 + 4\,(4a^2b)\,(-3abc)^3 + (-3abc)^4,$$

or performing the operations indicated, we have

$$(4a^2b - 3abc)^4 = 256a^8b^4 - 768a^7b^4c + 864a^6b^4c^2 \ldots$$
$$- 432a^5b^4c^3 + 81a^4b^4c^4.$$

The terms produced by this development are alternately positive and negative. This, it is evident, should always be the case, when the second term of the proposed binomial has the sign $-$.

133. The powers of any polynomial whatever may be found by the binomial theorem. Let it be proposed to find for example, the third power of the trinomial $a+b+c$.

In order to apply the rule to this case, we put $a+b=m$; the proposed is then reduced to the binomial $m+c$, and we have

$$(m+c)^3 = m^3 + 3m^2c + 3mc^2 + c^3$$

whence, restoring the value of m, we have

$$\begin{aligned}(a+b+c)^3 = a^3 &+ 3a^2b + 3ab^2 + b^3 \\ &+ 3a^2c + 6abc + 3b^2c \\ &+ 3ac^2 + 3bc^2 \\ &+ c^3.\end{aligned}$$

The same process, it is easy to see, may be applied to any polynomial whatever.

MISCELLANEOUS EXAMPLES.

1. To find the third power of $2a - b + c^2$.
2. To find the seventh power of $3a^5 - 2a^2$.
3. To find the fifth power of $a^2 - c - 2d$.
4. To find the third power of $2a^2 - 4ab + 3b^2$.

SECTION XVII.—Roots of Compound Quantities.

134. We pass next to the extraction of the roots of compound quantities, beginning with the third or cube root of numbers.

In the following table, we have the nine first numbers, with their third powers or cubes written under them respectively.

1,	2,	3,	4,	5,	6,	7,	8,	9,
1,	8,	27,	64,	125,	216,	343,	512,	729.

By inspection of this table, it will be perceived, that among numbers consisting of two or three figures, there are nine only, which are perfect third powers, the others have each for a root an entire number plus a fraction.

If the proposed number consists of not more than three figures, its third root or that of the greatest third power contained in it, may be found immediately by the above table.

Let it be proposed to extract the third root of a number, consisting of more than three figures, 103823, for example.

The proposed being comprised between 1000, the third power of 10, and 1000000, the third power of 100, its root will consist of two places, units and tens. To return therefore from the proposed to its root, let us observe the manner, in which the

units and tens of a number are employed in forming the third power of this number. For this purpose designating the tens by a and the units by b, we have

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3.$$

From this we learn, that the third power of a number consisting of units and tens, is composed of *the third power of the tens, the triple product of the square of the tens by the units, the triple product of the tens by the square of the units, and the third power of the units.*

If then we can determine in the proposed the third power of the tens, the tens of the root will be found by extracting the third root of this part. The third power of the tens, it is evident, can have no significant figure below the fourth place, the three figures on the right will, therefore, form no part of the third power of the tens, and may on this account be separated from the rest by a comma. The third power of the tens will then be contained in 103, the part at the left of the comma. The greatest third power contained in 103 is 64, the root of which is 4; 4 is, therefore, the significant figure in the tens of the root sought. Indeed, the proposed is evidently comprised between 64000, the third power of 40 or 4 tens, and 125000 the third power of 50 or 5 tens. The root sought is, therefore, composed of 4 tens and a certain number of units less than ten.

The tens of the root being thus obtained, we subtract the third power 64 from 103, the part of the proposed at the left of the comma, and to the remainder bring down the figures at the right. The result of this operation, 39823, must contain, from what has been said, *the triple product of the square of the tens by the units*, together with the two remaining parts in the third power of the root sought.

The square of the tens, it is evident, will contain no significant figure less than hundreds; on this account we separate 23, the two figures on the right of the remainder 39823, from the rest by a comma; 398, the figures on the left of the comma, will then contain the triple product of the square of the tens of the

root sought by the units and something more, in consequence of the hundreds arising from the two remaining parts of the third power of the root sought. Dividing therefore 398 by 48, the triple product of the square of the tens, already found, the quotient 8 will be the unit figure sought, or, from what has been said, it may be too large by 1 or 2.

To determine whether 8 be the right unit figure we raise 48 to the third power. This gives 110592, a number greater than the proposed; 8 is, therefore, too large for the unit figure. We next try 7; 47 raised to the third power gives 103823. The proposed is, therefore, a perfect third power, the root of which is 47.

The operation may be exhibited as follows.

$$\begin{array}{r|l}
103,823 & 47 \\
64 \\
\hline
398,23 & 48 \\
\hline
103,823 \\
\hline
\end{array}$$

Any number however large may be considered as composed of units and tens; the process for finding the root may therefore be reduced to that of the preceding example.

Let it be proposed, for example, to find the third root of 43725678. Considering the root of this number as composed of units and tens, 678 the three right hand figures, it is evident, will form no part of the third power of the tens. On this account we separate them from the rest by a comma. The third power of the tens being contained, then, in the part at the left of the comma, we obtain the tens of the root sought by extracting the third root of this part. Considering therefore, for the moment, the part of the proposed 43725 as a separate number, its third root, it is evident, may be found as in the preceding example. Performing the operations, we have 35 for the root and a remainder of 850. There will therefore be

35 tens in the root of the proposed, and in order to find the units, we bring down the three right hand figures 678 by the side of 850, which gives 850678. Separating next the two right hand figures of this last from the rest by a comma, and dividing the part on the left by the triple square of the tens already found, we obtain 2 for the unit figure of the root sought. To determine whether this is the right figure, we raise 352 to the third power, which gives 43614208, a result less than the proposed. 352 is, therefore, the root of the proposed to within less than a unit.

The operation may be exhibited as follows:

	43,725,678	352	
	27		
1s Dividend	167,25	27	1st Divisor
Third power of 35	42875		
2d Dividend	8506,78	3675,	2d Divisor
Third power of 352	43614208		
Remainder	111470		

The same process, it is easy to see, may be extended to any number however large. The rule, therefore, for the extraction of the third root will be readily inferred.

If it happens, that the divisor is not contained in the dividend prepared as above, a zero must be placed in the root, and the next figure brought down to form the dividend.

EXAMPLES.

1. Find the third root of 150568768. Ans. 532.
2. Find the third root of 205483447701. Ans. 5901.
3. Find the third root of 32977340218432. Ans. 32068.

135. If the proposed be a fraction its third root is found by extracting the third root of the numerator and denominator.

Thus $\sqrt[3]{\frac{8}{27}}$ is $\frac{2}{3}$.

If the denominator is not a perfect third power, it may be made so, by multiplying both terms by the square of the denominator; thus if the proposed be $\frac{3}{7}$, we multiply both terms by 49; the fraction then becomes $\frac{147}{343}$, the root of which is nearest $\frac{5}{7}$, accurate to within less than $\frac{1}{7}$.

136. We have seen, art. 94, that the square root of an entire number, which is not a perfect square, cannot be exactly assigned. The same is true with respect to the roots of all entire numbers, which are not perfect powers of a degree denoted by the index of the root.

The third root of a number which is not a perfect third power may be approximated by converting the number into a fraction, the denominator of which is a perfect third power. Thus let it be required to find the approximate third root of 15. This number may be put under the form $\frac{15 \times 12^3}{12^3} = \frac{25920}{1728}$, the third root of which is $\frac{29}{12}$, or $2\frac{5}{12}$, accurate to within less than $\frac{1}{12}$. If a greater degree of accuracy were required, we should convert the proposed into a fraction, the denominator of which is the third power of some number greater than 12.

In such cases it is most convenient to convert the proposed number into a fraction, the denominator of which shall be the third power of 10, 100, 1000, &c.

Thus if it be required to find the third root of 25 to within .001, we convert the proposed into a decimal, the denominator of which is the third power of 1000, viz. 25.000000000, the third root of which is 2.920 to within .001; we have then $\sqrt[3]{25} = 2.920$, accurate to within less than .001.

To approximate therefore the third root of an entire number by means of decimals, *we annex to the proposed three times as many zeros as there are decimal places required in the root, we then*

extract the root of the number thus prepared to within a unit, and point off for decimals, as many places as there are decimal figures required in the root.

137. If the proposed number contain decimals, beginning at the place of units, we separate the number both to the right and left into periods of three figures each, annexing zeros, if necessary, to complete the right hand period in the decimal part. We then extract the root, and point off for decimals in the root as many places as there are periods in the decimal part of the power.

If the proposed be a vulgar fraction, the most simple method of finding the third root is to convert the proposed into a decimal, the number of places in which shall be equal to three times the number of decimal figures required in the root. The question is thus reduced to extract the third root of a decimal fraction.

EXAMPLES.

1. Fnd the approximate third root of 79. Ans. 4.2908.
2. Find the approximate third root of $\frac{14}{25}$. Ans. 0.824.
3. Find the approximate third root of 3.00415. Ans. 1.4429.
4. Find the approximate third root of $15\frac{5}{8}$. Ans. 2.502.

138. By processes altogether similar to that, which we have employed in the extraction of the third root of numbers, we may extract the root of any degree whatever. The method of extracting the root of any degree whatever, in the case of algebraic quantities, is also founded upon the same principles. The following example will be sufficient to illustrate the course to be pursued, whatever the degree of the root required may be.

Let it be proposed to extract the fifth root of the polynomial

$$32a^{10}-80a^8b^3+80a^6b^6-40a^4b^9+10a^2b^{12}-b^{15}.$$

The proposed being arranged with reference to the powers of the letter a, we seek the fifth root of the first term $32a^{10}$.

Its root $2a^2$ will be the first term of the root sought. We write, therefore, $2a^2$ in the place of the quotient in division, and subtracting its fifth power from the whole quantity, we have for a remainder

$$-80a^8b^3+80a^6b^6+ \&c.$$

The second term of the binomial $(a+b)^5$ is $5a^4b$; this shows, that in order to obtain the second term of the root, we must divide $-80a^8b^3$, the second term of the proposed, by five times the 4th power of $2a^2$, the term of the root already found. Performing the operation we obtain $-b^3$. This will be, therefore, the second term of the root. Raising $2a^2-b^3$ to the fifth power, it produces the quantity proposed. The root is therefore obtained exactly. If the root contained more than two terms, it would be necessary to subtract the fifth power of $2a^2-b^3$ from the proposed quantity, and then in order to find the next term of the root, to divide the first term of the remainder by five times the 4th power of $2a^2-b^3$. In this case, however, only the first term of the divisor would be used; we should have therefore the same divisor, that was used the first time.

139. When the index of the root has divisors the root may be found more readily than by the general method. Thus the fourth root may be found by extracting the square root twice successively; for the square root of a^4 is a^2, and that of a^2 is a, the fourth root of a^4. In general, all roots of a degree marked by 4, 8 or any power of 2 may be found by successive extractions of the square root. Roots, the indices of which are not prime numbers, may be reduced to others of a degree less elevated. The 6th root, for example, may be found by first extracting the square and then the third root; for the square root of a^6 is a^3, and the third root of a^3 is a.

EXAMPLES.

1. To find the third root of $8x^3+36x^2+54x+27$.

Ans. $2x+3$.

2. To find the third root of $x^6 + 6x^5 - 40x^3 + 96x - 64$.
Ans. $x^2 + 2x - 4$.

3. To find the third root of $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.
Ans. $x^2 - 2x + 1$.

4. To find the third root of $27x^6 - 54x^5 + 63x^4 - 44x^3 + 21x^2 - 6x + 1$.
Ans. $3x^2 - 2x + 1$.

5. To find the fourth root of $16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4$.
Ans. $2a - 3x$.

SECTION XVIII.—Calculus of Radical Expressions.

140. Radical expressions, the roots of which cannot be found exactly, frequently occur in the solution of questions. On this account mathematicians have been led to investigate rules for performing upon quantities subjected to the radical sign, the operations designed to be performed upon their roots. In this way the calculations required in the solution of a question are frequently rendered more simple, and the extraction of the root is left to be performed at last, when the radical expression is reduced to the most simple form, which the nature of the question will allow.

ADDITION AND SUBTRACTION.

141. Radical expressions of the same degree, and which have the quantities placed under the radical sign also the same, are said to be *similar*.

The addition and subtraction of similar radicals is performed upon the coefficients. Thus the sum of the radicals $3\sqrt[3]{b}$, $9\sqrt[3]{b}$, is $12\sqrt[3]{b}$; the sum of $a\sqrt[5]{b^2c}$, $b\sqrt[5]{b^2c}$, $-c\sqrt[5]{b^2c}$ is $(a + b - c)\sqrt[5]{b^2c}$.

In like manner $9\sqrt[3]{a^4c}$, subtracted from $12\sqrt[3]{a^4c}$ gives

$3\sqrt[3]{a^4c}$, and $b\sqrt[7]{ab^2}$ subtracted from $a\sqrt[7]{ab^2}$ gives

$$(a-b)\sqrt[7]{ab^2}.$$

Radical expressions, which are at first dissimilar, frequently become similar when reduced to their most simple form. Thus, let it be required to add $5\sqrt[3]{2a^5b^2}$ and $a\sqrt[3]{54a^2b^2}$. These expressions, reduced to their most simple form, become $5a\sqrt[3]{2a^2b^2}$, $3a\sqrt[3]{2a^2b^2}$; their sum is therefore

$$8a\sqrt[3]{2a^2b^2}.$$

The addition and subtraction of dissimilar radicals can be effected only by means of the signs $+$ and $-$.

EXAMPLES.

1. To find the sum of $5a^2\sqrt{bc^2}$, and $a\sqrt{9a^2bc^2}$.
2. To find the sum of $ac^2\sqrt{16b^7c}$, and $3\sqrt{a^2b^7c^5}$.
3. To find the sum of $\sqrt{147a^2bc^5}$, and $5c^2\sqrt{75a^2bc}$.
4. To find the sum of $2\sqrt{8}$, $-7\sqrt{18}$, $5\sqrt{72}$, and $-\sqrt{50}$.
   Ans. $8\sqrt{2}$.
5. To find the sum of $8\sqrt{\frac{3}{4}}$, $-\frac{1}{2}\sqrt{12}$, $4\sqrt{27}$, and $-2\sqrt{\frac{3}{16}}$.
   Ans. $\frac{29}{2}\sqrt{3}$.
6. To find the sum of $2\sqrt{\frac{5}{3}}$, $\sqrt{60}$, $-\sqrt{15}$, and $\sqrt{\frac{3}{5}}$.
   Ans. $\frac{28}{15}\sqrt{15}$.
7. To find the sum of $\sqrt{18a^5b^3}$, and $\sqrt{50a^3b^3}$.
   Ans. $(3a^2b+5ab)\sqrt{2ab}$.
8. To subtract $\sqrt{9a^2b^2c}$ from $7a\sqrt{b^2c}$.
9. To subtract $\sqrt[3]{24}$ from $\sqrt[3]{192}$. Ans. $2\sqrt[3]{3}$.
10. To subtract $\sqrt[3]{\dfrac{a^2x}{2b}}$ from $\sqrt[3]{\dfrac{27a^5x}{2b}}$.
   Ans. $(3a-1)\sqrt[3]{\dfrac{a^2x}{2b}}$.

MULTIPLICATION AND DIVISION.

142. Let it be required to multiply $\sqrt[7]{a}$ by $\sqrt[7]{b}$, we have $\sqrt[7]{a} \times \sqrt[7]{b} = \sqrt[7]{ab}$; for $\sqrt[7]{a} \times \sqrt[7]{b}$ raised to the seventh power gives ab for the result, and $\sqrt[7]{ab}$ raised to the seventh power gives also ab for the result; whence the seventh powers of these expressions being equal, the expressions themselves must be equal.

The same reasoning may be applied to all similar cases; we have, therefore, the following rule for the multiplication of radical expressions of the same degree, viz. *Take the product of the quantities under the radical sign, observing to place the result under a sign of the same degree.*

Let it next be required to divide $\sqrt[5]{a}$ by $\sqrt[5]{b}$. In this case we have $\dfrac{\sqrt[5]{a}}{\sqrt[5]{b}} = \sqrt[5]{\dfrac{a}{b}}$; for the expressions $\dfrac{\sqrt[5]{a}}{\sqrt[5]{b}}$, $\sqrt[5]{\dfrac{a}{b}}$ being raised to the fifth power give each $\dfrac{a}{b}$; these expressions are therefore equal.

We have then the following rule for the division of one radical quantity by another of the same degree, viz. *Take the quotient arising from the division of the quantities under the radical sign, recollecting to place it under a sign of the same degree.*

EXAMPLES.

1. Multiply $\sqrt[3]{4}$, $7\sqrt[3]{6}$, and $\frac{1}{2}\sqrt[3]{5}$ together.

   Ans. $\frac{7}{2}\sqrt[3]{120}$.

2. Multiply $5\sqrt{3}$, $7\sqrt{\frac{8}{3}}$, and $\sqrt{2}$ together. Ans. 140.

3. Multiply $7 + 2\sqrt{6}$ by $9 - 5\sqrt{6}$. Ans. $3 - 17\sqrt{6}$.

4. Multiply $8 + 2\sqrt{7}$ by $8 - 2\sqrt{7}$. Ans. 36.

5. Multiply $5\sqrt{3} - 7\sqrt{6}$ by $2\sqrt{8} - 3$.

   Ans. $41\sqrt{6} - 71\sqrt{3}$.

6. Divide $\sqrt{243}$ by $\sqrt{48}$. Ans. $2\frac{1}{4}$.

7. Divide $\sqrt[3]{24a^5b^3}$ by $\sqrt[3]{81a^2b}$. Ans. $\frac{2}{3}a\sqrt[3]{b}$.

8. Divide 1 by $1+\sqrt{2}$. Expressing the quotient in the form of a fraction, and multiplying both terms by $1-\sqrt{2}$ we have Ans. $\sqrt{2}-1$.

9. Divide $1+\sqrt{6}$ by $2\sqrt{2}-\sqrt{3}$. Ans. $\sqrt{2}+\sqrt{3}$.

FORMATION OF POWERS AND EXTRACTION OF ROOTS.

143. Let it be required to raise the radical $\sqrt[5]{a^2b}$ to the third power; we have

$$(\sqrt[5]{a^2b})^3=\sqrt[5]{a^2b}\times\sqrt[5]{a^2b}\times\sqrt[5]{a^2b}=\sqrt[5]{a^6b^3}$$

according to the rule established for multiplication.

Whence to raise a radical quantity to any power; *we raise the quantity placed under the radical sign to the power required, observing to place the result under the same radical sign.*

When the index of the radical is a multiple of the exponent of the power to which the radical is to be raised, it may be raised to the power required in a more simple manner than by the preceding rule.

Thus let it be required to raise $\sqrt[4]{2a}$ to the second power. The proposed from what has been said, art. 139, may be put under the form $\sqrt{\sqrt{2a}}$; but to raise this expression to the second power, it is sufficient to suppress the first radical sign; whence $(\sqrt[4]{2a})^2=\sqrt{2a}$.

Again, let it be required to raise $\sqrt[9]{5b}$ to the third power. The proposed may be put under the form $\sqrt[3]{\sqrt[3]{5b}}$; whence $(\sqrt[9]{5b})^3=\sqrt[3]{5b}$.

Whence if the index of the radical is divisible by the exponent

of the power, to which the proposed quantity is to be raised, the operation is performed by dividing the index of the radical by the exponent of the power.

144. With respect to the extraction of roots, it is evident, from the preceding rules, that to extract the root of a radical, *we may extract the root of the quantity placed under the radical sign, the result being left under the same radical sign, or we may multiply the index of the radical by the index of the root to be extracted.*

Thus, $\sqrt[3]{\sqrt[4]{27a^3}}=\sqrt[4]{3a}$. $\sqrt[3]{\sqrt[4]{3c}}=\sqrt[12]{3c}$.

EXAMPLES.

1. To raise $\sqrt[3]{a^4b^2}$ to the fourth power.
2. To raise $\sqrt[12]{a^3b^2c}$ to the sixth power.
3. To find the fourth root of $\sqrt[3]{32a^4b^8}$.
4. To find the fifth root of $\sqrt[4]{243a^{10}b^5}$.

REDUCTION OF RADICAL EXPRESSIONS TO THE SAME INDEX.

145. It follows from the principles established above, that *if we multiply at the same time the index of the radical and the exponents of the quantity placed under the radical sign by the same number, the value of the radical remains the same.*

Thus if we multiply the index of the radical $\sqrt[5]{a^2b}$ by 3, we have $\sqrt[15]{a^2b}$, the third root of the proposed; if then we multiply the exponent of the quantity placed under the radical sign by 3, we have $\sqrt[15]{a^6b^3}$ the third power of $\sqrt[15]{a^2b}$; the second operation, therefore, restores the expression to its original value.

146. By means of this last principle, we may reduce two or

more radicals of different indices to the same index. Thus let there be the two radicals $\sqrt[3]{2a}$, $\sqrt[4]{b^2c}$. Multiplying the index and also the exponents of the quantities placed under the radical sign in the first by 4, and in the second by 3, we have for the first $\sqrt[12]{2^4a^4}$ or $\sqrt[12]{16a^4}$, and for the second $\sqrt[12]{b^6c^3}$. The proposed are, therefore, reduced to equivalent expressions having a common index 12.

In like manner, the three quantities

$$\sqrt[3]{ab^2},\ \sqrt[5]{a^2b^3},\ \sqrt[7]{c^4d^3},$$

become respectively

$$\sqrt[105]{a^{35}b^{70}},\ \sqrt[105]{a^{42}b^{63}},\ \sqrt[105]{c^{60}d^{45}}$$

having a common index 105.

From what has been done we have the following rule for reducing radical expressions to the same index, viz. *Multiply at the same time the index belonging to each radical sign, and the exponents of the quantities placed under this sign, by the product of the indices belonging to all the other radical signs.*

If the indices of the radicals have common factors, the calculations are rendered more simple, by taking for the common index the least number exactly divisible by each of the indices.

A quantity, which has no radical sign, may on the same principles be placed under a radical sign; for this purpose, *we raise the quantity proposed to the power denoted by the index of the radical sign, under which it is to be placed.*

Thus if it be required to put the quantity a^2 under the sign $\sqrt[5]{\ }$, we have for the result $\sqrt[5]{a^{10}}$.

147. Radical expressions having different indices must be reduced to the same index before applying to them the rules for multiplication and division laid down above. The following examples will serve as an additional exercise in the multiplication and division of radical quantities.

1. Multiply $\sqrt[4]{2}$ by $\sqrt[5]{3}$. Ans. $\sqrt[20]{2592}$.
2. Multiply $\sqrt{a}$ by $\sqrt[3]{b}$. Ans. $\sqrt[6]{a^3b^2}$.
3. Multiply $\sqrt[3]{a}$ by $\sqrt[4]{b}$. Ans. $\sqrt[12]{a^4b^3}$.
4. Multiply $3a\sqrt[4]{8a^2}$ by $2b\sqrt[4]{4a^2c}$. Ans. $12a^2b\sqrt[4]{2c}$.
5. Multiply $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[4]{5}$ together. Ans. $\sqrt[12]{648000}$.
6. Multiply $\sqrt[7]{\frac{4}{3}}$, $\sqrt[3]{\frac{1}{2}}$, and $\sqrt[14]{6}$ together. Ans. $\sqrt[42]{\frac{2}{27}}$.
7. Multiply $2\sqrt{6}-3\sqrt{5}$ by $4\sqrt{3}-\sqrt{10}$.
   Ans. $39\sqrt{2}-16\sqrt{15}$.
8. Multiply $4\sqrt{\frac{7}{3}}+5\sqrt{\frac{1}{2}}$ by $\sqrt{\frac{7}{3}}+2\sqrt{\frac{1}{2}}$.
   Ans. $\frac{43}{3}+\frac{13}{6}\sqrt{42}$.
9. Divide $\sqrt[4]{6}$ by $\sqrt[3]{2}$. Ans. $\frac{1}{2}\sqrt[12]{55296}$.
10. Divide $\sqrt[5]{135}$ by $\sqrt{3}$. Ans. $\sqrt[10]{75}$.
11. Divide $a\sqrt[3]{b}$ by $\sqrt[4]{b}$. Ans. $a\sqrt[12]{b}$.
12. Divide $\sqrt{a}$ by $\sqrt{b}+\sqrt{c}$. Ans. $\dfrac{\sqrt{ab}-\sqrt{ac}}{b-c}$.
13. Divide $\sqrt{3}$ by $1+\sqrt{2}$. Ans. $\sqrt{3}(\sqrt{2}-1)$.
14. Divide $\frac{1}{2}\sqrt{\frac{1}{2}}$ by $\sqrt{2}+3\sqrt{\frac{1}{2}}$. Ans. $\frac{1}{10}$.
15. Divide 1 by $\sqrt[n]{a+\sqrt{b}}$. Ans. $\sqrt[n]{\dfrac{a-\sqrt{b}}{a^2-b}}$.
16. Divide $\sqrt[n]{\sqrt{a}+\sqrt{b}}$ by $\sqrt[n]{\sqrt{a}-\sqrt{b}}$.

   Ans. $\sqrt[n]{\dfrac{a+b+2\sqrt{ab}}{a-b}}$.

SECTION XIX.—Theory of Exponents.

148. We have seen, art. 28, that with respect to the same letter, division is performed by subtracting the exponent of

the divisor from that of the dividend. The application of this rule to the case, in which the exponent of the divisor is equal to that of the dividend, gives rise to the exponent 0. An expression a^0, in which this exponent is found, is to be regarded, art. 31, as *a symbol equivalent to unity.*

149. The application of the same rule to the case, in which the exponent of the divisor exceeds that of the dividend, gives rise to *negative* exponents. Thus let it be required to divide a^3 by a^5. Subtracting the exponent of the latter from that of the former, we have a^{-2} for the result. But a^3 divided by a^5 is expressed by the fraction $\frac{a^3}{a^5}$; reducing this fraction to its lowest terms, we have $\frac{1}{a^2}$. The expression a^{-2} must therefore be regarded as equivalent to $\frac{1}{a^2}$.

In like manner $\frac{a^m}{a^{m+n}}$ gives by subtracting the exponent of the divisor from that of the dividend a^{-n}; but the fraction $\frac{a^m}{a^{m+n}}$ gives when reduced to its lowest terms $\frac{1}{a^n}$; whence a^{-n} is equivalent to $\frac{1}{a^n}$.

The expression a^{-n} is, therefore, the symbol of a division, which cannot be performed. *Its true value is the quotient of unity divided by* a *raised to a power denoted by the negative exponent* n.

150. To find the roots of monomials, we divide, art. 122, the exponents of the proposed by the index of the root required. The application of this rule to the case, in which the exponents of the proposed are not divisible by the index of the root, gives rise to *fractional* exponents. Thus let the third root of a be required. Indicating upon the exponent of a the operation required in order to obtain the third root, we have for the result $a^{\frac{1}{3}}$. But we have agreed to indicate the third root by $\sqrt[3]{}$

the expressions $\sqrt[3]{a}$, $a^{\frac{1}{3}}$ are, therefore, to be regarded as equivalent. In like manner, we have

$$a^{\frac{1}{2}} = \sqrt{a},\ a^{\frac{2}{3}} = \sqrt[3]{a^2},\ a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

The expression $a^{\frac{m}{n}}$ *is, therefore, to be regarded, as a symbol equivalent to the* n*th root of the* m*th power of* a.

151. The two preceding cases sometimes meet in the same expression. This gives rise to *negative* fractional exponents. Thus let it be required to extract the seventh root of a^3 divided by a^5; we have $\frac{a^3}{a^5} = a^{-2}$, the seventh root of which is $a^{-\frac{2}{7}}$. In like manner the nth root of

$$\frac{a^n}{a^{m+n}} = a^{-\frac{m}{n}}$$

The expression $a^{-\frac{m}{n}}$ is, therefore, the symbol of a division which cannot be performed, combined with the extraction of a root. *Its true value is the* n*th root of the quotient of unity divided by* a *raised to the* m*th power.*

The expressions a^0, a^{-m}, $a^{\frac{m}{n}}$, $a^{-\frac{m}{n}}$, derived in the manner above explained from rules previously established, have become by agreement notations equivalent respectively to 1, $\frac{1}{a^m}$, $\sqrt[n]{a^m}$, $\sqrt[n]{\frac{1}{a^m}}$; we may, therefore, at pleasure substitute the former of these expressions for the latter, and the converse.

152. We proceed to show, that the rules already established for performing the operations of arithmetic upon quantities affected with entire and positive exponents are sufficient for these operations, whatever the exponents may be, with which the quantities are affected.

MULTIPLICATION.

Let it be required to multiply $a^{\frac{3}{5}}$ by $a^{\frac{2}{3}}$. To perform the operation required, it is sufficient *to add the exponents.*

Indeed $a^{\frac{3}{5}} = \sqrt[5]{a^3}$, $a^{\frac{2}{3}} = \sqrt[3]{a^2}$, whence

$$a^{\frac{3}{5}} \times a^{\frac{2}{3}} = \sqrt[5]{a^3} \times \sqrt[3]{a^2} = \sqrt[15]{a^{19}} = a^{\frac{19}{15}}.$$

But adding the exponents, we have

$$a^{\frac{3}{5}} \times a^{\frac{2}{3}} = a^{\frac{3}{5} + \frac{2}{3}} = a^{\frac{19}{15}},$$

the same result as before.

Again, let it be required to multiply $a^{-\frac{3}{4}}$ by $a^{\frac{5}{6}}$; we have

$$a^{\frac{5}{6}} = \sqrt[6]{a^5}, \text{ and } a^{-\frac{3}{4}} = \sqrt[4]{\frac{1}{a^3}},$$

whence $a^{-\frac{3}{4}} \times a^{\frac{5}{6}} = \sqrt[4]{\frac{1}{a^3}} \times \sqrt[6]{a^5} = \sqrt[12]{\frac{1}{a^9}} \times \sqrt[12]{a^{10}}$.

$$= \sqrt[12]{a} = a^{\frac{1}{12}}.$$

But adding the exponents of the proposed, we have

$$a^{\frac{5}{6}} \times a^{-\frac{3}{4}} = a^{\frac{5}{6} - \frac{3}{4}} = a^{\frac{1}{12}}$$

the same result as by the former operation.

Let it be required next to multiply $a^{-\frac{m}{n}}$ by $a^{\frac{p}{q}}$;

we have $a^{-\frac{m}{n}} = \sqrt[n]{\frac{1}{a^m}}$, $a^{\frac{p}{q}} = \sqrt[q]{a^p}$;

whence $a^{-\frac{m}{n}} \times a^{\frac{p}{q}} = \sqrt[n]{\frac{1}{a^m}} \times \sqrt[q]{a^p}$

$$= \sqrt[nq]{\frac{a^{np}}{a^{mq}}} = \sqrt[nq]{a^{np-mq}} = a^{\frac{np-mq}{nq}}.$$

We arrive at the same result by adding the exponents of the proposed.

Indeed $a^{-\frac{m}{n} + \frac{p}{q}} = a^{\frac{np-mq}{nq}}$.

To multiply two monomials therefore, it is sufficient, whatever the exponents, *to add the exponents of the letters, which are the same in each.*

EXAMPLES.

1. Multiply $a^{\frac{1}{3}}c^{\frac{1}{3}}$, $a^{-\frac{1}{2}}b$, and $c^{\frac{3}{4}}b^{-\frac{1}{2}}$ together.
Ans. $a^{-\frac{1}{6}}b^{\frac{1}{2}}c^{\frac{13}{12}}$.

2. Multiply $\dfrac{a}{b^{\frac{1}{2}}c^{\frac{3}{4}}}$ by $\dfrac{a^{\frac{7}{8}}b}{c^{-\frac{1}{2}}}$. Ans. $a^{\frac{15}{8}}b^{\frac{1}{2}}c^{-\frac{1}{4}}$.

3. Multiply $a^{\frac{3}{4}}+b^{\frac{2}{5}}$ by $a^{\frac{3}{4}}-b^{\frac{2}{5}}$. Ans. $a^{\frac{3}{2}}-b^{\frac{4}{5}}$.

4. Multiply $3+5^{\frac{1}{2}}$ by $2-5^{\frac{1}{2}}$. Ans. $1-5^{\frac{1}{2}}$.

DIVISION.

153. Whatever the exponents may be, in order to divide one monomial by another, we subtract for each letter *the exponent of the divisor from that of the dividend.*

Indeed, since the exponent of each letter in the quotient should be such, that when added to the exponent of the same letter in the divisor, the sum will be equal to the exponent of the dividend, it follows, that the exponent of the quotient should be equal to the difference between that of the divisor and the dividend.

EXAMPLES.

1. Divide $a^{\frac{2}{3}}$ by $a^{-\frac{3}{4}}$. Ans. $a^{\frac{17}{12}}$.

2. Divide $a^{\frac{3}{4}}$ by $a^{\frac{4}{5}}$. Ans. $a^{-\frac{1}{20}}$.

3. Divide $a^{\frac{2}{5}}b^{\frac{3}{4}}$ by $a^{-\frac{1}{2}}b^{\frac{7}{8}}$. Ans. $a^{\frac{9}{10}}b^{-\frac{1}{8}}$.

4. Divide $a^{\frac{3}{4}}-b^{\frac{3}{4}}$ by $a^{\frac{1}{4}}-b^{\frac{1}{4}}$. Ans. $a^{\frac{1}{2}}+a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}}$.

5. Divide $5a^{\frac{25}{12}}-41a^{\frac{13}{12}}b+42a^{\frac{1}{12}}b^2$ by $5a^{\frac{7}{4}}-6a^{\frac{3}{4}}b$.
Ans. $a^{\frac{1}{3}}-7a^{-\frac{2}{3}}b$.

FORMATION OF POWERS AND EXTRACTION OF ROOTS.

154. From the rule for multiplication, it follows, that to raise a monomial to any power, it is necessary whatever the exponents of the letters, to multiply the exponent of each letter by the exponent of the power required.

Thus $a^{-\frac{2}{3}}$ raised to the third power

$$=a^{-\frac{2}{3}+(-\frac{2}{3})+(-\frac{2}{3})}=a^{-\frac{6}{3}}=a^{-2}.$$

Conversely, to extract the root of a monomial, *we divide the exponent of each letter by the index of the root.*

Thus $\sqrt[3]{a^{-2}}=a^{-\frac{2}{3}}$.

The utility of exponents, of the kind which we are here considering, consists principally in this, that the calculation of quantities affected with these exponents is performed by the rules already established for quantities affected with entire and positive exponents. The calculation is moreover reduced to operations upon fractions, with which we are already familiar.

155. By means of negative exponents we may give an entire form to fractional expressions. Thus, let there be the fraction $\frac{x}{y^2}$, this is the same as $x\times\frac{1}{y^2}$; but $\frac{1}{y^2}=y^{-2}$; whence $\frac{x}{y^2}=xy^{-2}$.

156. Fractional and negative exponents enable us to arrange polynomials, which contain radical terms. Thus let it be required to arrange the polynomial

$$2\sqrt{a}+\frac{1}{a}+\sqrt[3]{a^7}+\frac{4}{a\sqrt[3]{a}}+\frac{1}{\sqrt[3]{a^2}}+2\sqrt[6]{a^5}$$

according to the descending powers of the letter a.

To perform the operation required, 1°. we give to the radical quantities fractional exponents; 2°. we reduce to an entire form, terms which have denominators; 3°. we reduce all the exponents of the letter, according to which the arrangement is to be made, to their least common denominator. The proposed

may then be arranged according to the powers of the letter required.

In the preceding example we have for the result

$$a^{\frac{14}{6}}+2a^{\frac{5}{6}}+2a^{\frac{3}{6}}+a^{-\frac{4}{6}}+a^{-\frac{6}{6}}+4a^{-\frac{8}{6}}.$$

SECTION XX.—Proportions.

157. When two quantities are compared with respect to their magnitude, the result of the comparison is called their *ratio*. In general, there are two different ways, in which the magnitude of two quantities may be compared; 1°. we may wish to determine how much the greater exceeds the less; the result is then obtained by subtraction, and is called the ratio of the quantities *by difference;* 2°. we may wish to determine how often one of the quantities is contained in the other; the result is then found by division and is called the ratio of the quantities *by quotient.*

Thus, the ratio by difference of the quantities a and b is $a-b$, and the ratio by quotient is $\frac{a}{b}$; a and b are the *terms* of the ratio.

The same quantity may be added to, or subtracted from, both terms of a ratio by difference without changing the ratio, for

$$a-b=(a+c)-(b+c)=(a-c)-(b-c).$$

The two terms of a ratio by quotient may be multiplied or divided by the same quantity without changing the ratio, for

$$\frac{a}{b}=\frac{am}{bm}.$$

Ratios by difference are sometimes called *arithmetical* ratios and those by quotient *geometrical* ratios.

158. An expression for two equal ratios is called a *proportion.*

If the ratios are by difference, the proportion is called a *proportion by difference.* Thus the equality

$$b - a = d - c,$$

is a proportion by difference, and is usually written thus,

$$a\,.\,b : c\,.\,d.$$

If the ratios are by quotient, the proportion is called a *proportion by quotient.* Thus, the equality $\frac{a}{b} = \frac{c}{d}$ is a proportion by quotient, and is usually written

$$a : b :: c : d.$$

The proportions above are read thus, a is to b as c to d. The first and last terms are called the extremes of the proportion; the second and third are called the *means;* a is called the *antecedent,* b the *consequent* of the first ratio; c the *antecedent,* d the *consequent* of the second ratio.

Proportion by difference is sometimes called *arithmetical* proportion, that by quotient *geometrical proportion.* Proportion by difference is now, however, more commonly called equidifference, while the term proportion is limited to proportions by quotient.

EQUIDIFFERENCES.

159. Let there be the equidifference $a\,.\,b : c\,.\,d$; this is the same with the equation $b - a = d - c$, from which we deduce

$$a + d = b + c.$$

Thus in an equidifference *the sum of the extremes is equal to the sum of the means.* This is the leading property of equidifferences.

Reciprocally, let there be four quantities a, b, c, d, such that $a + d = b + c$. From this equation we obtain

$$b - a = d - c, \text{ or } a\,.\,b : c\,.\,d.$$

Thus, *if there be four quantities such, that any two of them give the same sum with the other two, the first are the extremes, the second the means.* or the converse, *of an equidifference.*

Any three terms of an equidifference are sufficient to determine the fourth; thus, from the equidifference, $a \,.\, b : c \,.\, d$, we deduce $a = b - d + c$, $b = a + d - c$.

In the equidifference $a \,.\, b : c \,.\, d$, let $c = b$; we have

$$a \,.\, b : b \,.\, d.$$

This is called a *continued* equidifference, and b is called an *arithmetical mean* between a and d.

From the equidifference $a \,.\, b : b \,.\, d$, we deduce

$$b = \tfrac{1}{2}(a + d);$$

the arithmetical mean between two quantities is, therefore, equal to half their sum.

160. In order that an equidifference may exist, it is sufficient, that the sum of the extremes should be equal to the sum of the means; we may, therefore, make any transposition of the terms, of an equidifference, which will not alter the equality between the sum of the extremes and that of the means. The equation $a - b = c - d$ furnishes the eight following equidifferences,

$$a \,.\, b : c \,.\, d,\quad a \,.\, c : b \,.\, d,\quad d \,.\, b : c \,.\, a,\quad d \,.\, c : b \,.\, a,$$
$$b \,.\, a : d \,.\, c,\quad b \,.\, d : a \,.\, c,\quad c \,.\, d : a \,.\, b,\quad c \,.\, a : d \,.\, b.$$

PROPORTION BY QUOTIENT.

161. Let us take the proportion $a : b :: c : d$; this returns to $\frac{b}{a} = \frac{d}{c}$, an equation, which gives

$$ad = bc.$$

In a proportion by quotient, therefore, *the product of the extremes is equal to the product of the means.* This is the fundamental property of proportions.

Reciprocally, let there be four quantities a, b, c, d, such, that $ad = bc$; this leads to the equation $\frac{b}{a} = \frac{d}{c}$, or

$$a : b : c : d.$$

Whence *if four quantities be such, that any two of them give the same product as the remaining two, the first will form the extremes and the second the means*, or the converse, *of a proportion.*

Three terms of a proportion are sufficient to determine the fourth; thus from the proportion $a:b::c:d$, we deduce

$$a=\frac{bc}{d},\ b=\frac{ad}{c},\ \&c.$$

The proportion $a:b::b:d$, in which the two mean terms are the same, is called a *continued* proportion, and b is called a *mean proportional* between a and d.

From the continued proportion $a:b::b:d$, we deduce $b^2=ad$, whence $b=\sqrt{ad}$. Thus *to find a mean proportional between two quantities, we take the square root of their product.*

162. In order that a proportion may exist, it is sufficient, that the product of the extremes should be equal to that of the means. We may, therefore, make any transposition in the terms of a proportion, which will leave the product of the extremes equal to that of the means. Thus the equation $\frac{b}{a}=\frac{d}{c}$ gives the eight following proportions

$$a:b::c:d,\quad a:c::b:d,\quad b:d::a:c,\quad d:c::b:a,$$
$$b:a::d:c,\quad c:a::d:b,\quad d:b::c:a,\quad c:d::a:b.$$

163. The same quantity m, it is evident, may be added to or subtracted from the equation $\frac{b}{a}=\frac{d}{c}$, so that we have

$$\frac{b}{a}\pm m=\frac{d}{c}\pm m;$$

whence
$$\frac{b\pm ma}{a}=\frac{d\pm mc}{c},$$

but this last may assume the form

$$\frac{c}{a}=\frac{d\pm mc}{b\pm ma};$$

from which we have the proportion

$$b\pm ma:d\pm mc::a:c;$$

but since $\frac{c}{a}=\frac{d}{b}$, we have also

$$\frac{d}{b}=\frac{d\pm mc}{b\pm ma},$$

from which we have the proportion

$$b \pm ma : d \pm mc :: b : d.$$

These two proportions may be enunciated thus; *The first consequent plus or minus its antecedent taken a given number of times, is to the second consequent plus or minus its antecedent taken the same number of times, as the first term is to the third, or as the second is to the fourth.*

164. The expression $\dfrac{d \pm mc}{b \pm ma} = \dfrac{c}{a}$ returns to

$$\frac{d + mc}{b + ma} = \frac{c}{a},\ \frac{d - mc}{b - ma} = \frac{c}{a};$$

whence
$$\frac{d + mc}{b + ma} = \frac{d - mc}{b - ma},$$

or
$$b + ma : d + mc :: b - ma : d - mc$$

or, changing the relative places of the means,

$$b + ma : b - ma :: d + mc : d - mc;$$

whence making $m = 1$, we have

$$b + a : b - a :: d + c : d - c,$$

a proportion which may be enunciated thus,

The sum of the first two terms is to their difference, as the sum of the last two is to their difference.

165. The proportion $a : b :: c : d$ may be written thus,

$$a : c :: b : d,$$

we have then

$$\frac{c}{a} \pm m = \frac{d}{b} \pm m;$$

from which we obtain

$$c \pm ma : d \pm mb :: a : b, \text{ or } :: c : d;$$

whence, *the second antecedent plus or minus the first taken a given number of times, is to the second consequent plus or minus the first taken the same number of times, as any one of the antecedents whatever is to its consequent.*

If in the above proportion we make $m = 1$, we have

$$c \pm a : d \pm b :: a : b, \text{ or } :: c : d;$$

whence
$$c + a : c - a :: d + b : d - b.$$

Therefore, *the sum or difference of the antecedents is to the sum or difference of the consequents, as one antecedent is to its consequent; and the sum of the antecedents is to their difference, as the sum of the consequents is to their difference.*

166. Let there be the series of equal ratios

$$a : b :: c : d :: e : f :: g : h \dots .$$

or $$\frac{b}{a} = \frac{d}{c} = \frac{f}{e} = \frac{h}{g}.$$

Making $\frac{b}{a} = q$; we have

$$\frac{b}{a} = q, \frac{d}{c} = q, \frac{f}{e} = q, \frac{h}{g} = q;$$

whence $$b = aq, d = cq, f = eq, h = gq;$$

adding these equations member to member, we have

$$b + d + f + h = (a + c + e + g)q;$$

whence $$\frac{b + d + f + h}{a + c + e + g} = q = \frac{b}{a},$$

or $$a + c + e + g : b + d + f + h :: a : b;$$

whence *in a series of equal ratios, the sum of any number whatever of antecedents, is to the sum of the like number of consequents, as one antecedent is to its consequent.*

167. Let there be the two equations $\frac{b}{a} = \frac{d}{c}, \frac{f}{e} = \frac{h}{g}$; multiplying these equations, member to member, we have

$$\frac{bf}{ae} = \frac{dh}{cg},$$

or $$ae : bf :: cg : dh.$$

We obtain the same result by multiplying, term by term, the proportions $a : b :: c : d$, $e : f :: g : h$; this is called multiplying the proportions *in order*; it follows then. *that if two proportions be multiplied in order, the results will be proportional.*

It will be seen also, that if two proportions be divided, term by term, or in *order*, the quotients will be proportional.

If in the equation $\frac{b}{a}=\frac{d}{c}$ we raise both members to the mth power, we have

$$\frac{b^m}{a^m}=\frac{d^m}{c^m}$$

which gives $a^m : b^m :: c^m : d^m$.

It follows, therefore, *that the second, third, and in general the similar powers of four proportional quantities are also proportional.*

In like manner, it may be shown *that the roots of the same degree of four proportional quantities are also proportional.*

QUESTIONS IN WHICH PROPORTIONS ARE CONCERNED.

1. The sum of the squares of two numbers is to the difference of their squares as 17 to 8, and their product is 15. What are the numbers?

Putting x and y for the numbers, we have by the first condition $\quad x^2+y^2 : x^2-y^2 :: 17 : 8;$

whence art. 164, $\quad 2x^2 : 2y^2 :: 25 : 9,$

or $\quad x^2 : y^2 :: 25 : 9;$

whence art. 167, $\quad x : y :: 5 : 3,$

wherefore $\quad 3x=5y.$

By the second condition we have $xy=15$; comparing this with the equation just found, we readily obtain $x=5$, $y=3$.

2. The product of two numbers is 24, and the difference of their third powers is to the third power of their difference as 19 to 1. Required the numbers.

Let x and y be the numbers, we have by the question

$$xy=24$$

$$x^3-y^3 : (x-y)^3 :: 19 : 1.$$

Transposing terms and developing $(x-y)^3$ in this last, we have $\quad x^3-y^3 : 19 :: x^3-3x^2y+3xy^2-y^3 : 1;$

whence art. 165, $\quad 3x^2y-3xy^2 : 18 :: (x-y)^3 : 1,$

or dividing by $x-y$ and substituting for xy its value from the first equation $\quad 72 : 18 :: (x-y)^2 : 1;$

whence $\quad x-y=2.$

Comparing this last with the first equation we obtain

$$x = 6,\ y = 4.$$

3. The sum of two numbers is to their difference as 3 to 1, and the difference of their third powers is 56; what are the numbers? Ans. 4 and 2.

4. There are two numbers, whose product is 135, and the difference of their squares is to the square of their difference as 4 to 1. What are the numbers? Ans. 15 and 9.

5. A merchant mixes wheat, which cost him 10 shillings a bushel, with barley which cost him 4 shillings a bushel, in such proportion as to gain 43¾ per cent. by selling the mixture at 11 shillings a bushel. Required the proportion.

Ans. 14 bushels of wheat to 9 of barley.

6. The product of two numbers is 63, and the square of their sum is to the square of their difference as 64 to 1. What are the numbers? Ans. 9 and 7.

SECTION XXI.—Progressions.

168. A series of quantities increasing or decreasing by a constant difference, is called an *arithmetical progression* or *progression by difference.* The constant difference is called the *ratio* of the progression.

Thus let there be the two following series

$$1,\ 4,\ 7,\ 10,\ 13,\ 16$$

$$60,\ 56,\ 52,\ 48,\ 44,\ 40.$$

the first is called an *increasing* progression, the ratio of which is 3; the second is called a *decreasing* progression, the ratio of which is 4.

To indicate that the quantities $a, b, c, d \ldots$ form a progression by difference, we write them thus

$$\div\, a \,.\, b \,.\, c \,.\, d \ldots .$$

A progression by difference, it will be readily perceived, is

simply a series of continued equidifferences. Each term therefore is at once antecedent and consequent, with the exception of the first term, which is only an antecedent, and of the last, which is only a consequent. The progression $\div a . b . c . d$ is enunciated thus, a is to b as b to c, as c to d, &c.

169. Let us take the increasing progression

$$\div a . b . c . k . h \ldots$$

and let d represent the ratio.

From the nature of the progression, we have, it is evident,

$$b = a + d$$
$$c = a + 2d$$
$$k = a + 3d,$$

from which it is readily inferred, *that a term of any rank whatever is equal to the first term plus as many times the ratio, as there are units in the number of the preceding terms.*

Let L represent a term of any rank whatever, and let n denote the number, which marks the place of this term; we have from what has been said

$$L = a + (n - 1)d.$$

This expression for L is called the *general term* of the series. If the series were decreasing, we should have, as it is easy to see, for the general term

$$L = a - (n - 1)d.$$

By means of the above formulas we may find any term of a progression by difference, when the first term, the number of the term required, and the ratio are given.

Thus, let it be required to find the 50th term of the progression $\div 1 . 4 . 7 \ldots\ldots$, we have by the first formula

$$L = 1 + (50 - 1)3 = 148.$$

Again let it be required to find the 40th term of the progression $\div 5 . 3 . 1 \ldots$, we have by the second formula

$$L = 5 - (40 - 1)2 = -73.$$

170. The first and last terms of a progression are called the *extremes;* if the number of terms be odd, the middle term is called the *mean;* if the number of terms be even, the two

terms having the same number of terms on each opposite side are called the *means.*

Let us take the general progression $\div a . b . c \ldots . h . k . l,$ from the nature of the progression, we have

$$b - a = l - k$$

whence $\quad b + k = a + l$

so also $\quad c - b = k - h$

whence $\quad c + h = b + k = a + l,$

$$\ldots\ldots\ldots\ldots$$

from which we infer that in a progression by difference, *the sum of any two terms taken at equal distances from the extremes is equal to the sum of the extremes.*

Let S represent the sum of all the terms in the progression $\div a . b . c \ldots . h . k . l .$ Writing this progression in an inverse order below itself, we have

$$S = a + b + c \ldots . h + k + l$$
$$S = l + k + h \ldots . c + b + a;$$

adding these equations member to member and uniting the corresponding terms, we have

$$2S = (a + l) + (b + k) \ldots . + (h + c) + (k + b) + (l + a)$$

but the parts $b + k,\ h + c \ldots$ are equal each to $a + l;$ the number of these parts moreover is the same, it is evident, with the number of terms in the progression; designating then this number by n, we have

$$2S = n(a + l);$$

whence $\quad S = \dfrac{n(a+l)}{2}.$

By means of this formula we find the sum of all the terms, when the first term, the last term and the number of the terms are given.

EXAMPLES.

1. What is the sum of the natural series of numbers 1, 2, 3, 4, &c. up to 1000?

2. The last term in a progression by difference is 60, the first term 12, and the number of terms 5. What is the sum of all the terms?

3. What is the sum of the uneven numbers 1, 3, 5, 7, &c. up to 93?

171. The equations $L = a + (n-1)d$, $S = \frac{n(a+l)}{2}$ furnish us with the means of resolving the following general problem, viz. *Any three of the five quantities,* a, d, n, l *and* s, *which enter into a progression by difference, being given, to determine the remaining two.*

This general problem resolves itself into as many particular problems as there are combinations of 5 letters taken 2 and 2, or 3 and 3 at a time. The number will therefore be 10. See the enunciations below.

Let there be given 1°. a, d, n to find l and s
2°. a, d, l . . . n and s
3°. a, d, s . . . n and l
4°. a, n, l . . . d and s
5°. a, n, s . . . d and l
6°. a, l, s . . . d and n
7°. d, n, l . . . a and s
8°. d, n, s . . . a and l
9°. d, l, s . . . a and n
10°. n, l, s . . . a and d

172. Let it now be proposed to find the number of terms in the progression by difference, the sum of which is 145, the first term 1, and the ratio 3.

This example is a particular case of the third general problem; to prepare a formula for its solution, we eliminate L from the equations

$$L = a + (n-1)d, \qquad S = \frac{n(a+l)}{2},$$

by which we obtain for the equation for n

$$n^2 - \frac{(d-2a)}{d} n = \frac{2s}{d};$$

resolving this equation we have

$$n = \frac{d - 2a \pm \sqrt{(d-2a)^2 + 8ds}}{2d}.$$

from which we obtain 10 for the number of terms sought. This being done we readily obtain by means of the formula for L the last term required.

Let the learner prepare the formulas and solve the following particular cases.

1. To find the first and last terms in a progression by difference, the sum of which is 567, the number of terms 21 and the ratio 2.

2. The sum of a progression by difference is 1455, the first term 5, and the number of terms 30. What is the last term and the ratio?

3. The first term in a progression by difference is 5, the last term 185, and the ratio 6, to find the number of terms, and the sum of all the terms.

4. The first term of a progression by difference is 3, the last term 41, and the sum of all the terms 440. What is the num ber of terms and the ratio?

173. The formula $L = a + (n-1)d$ gives $d = \frac{l-a}{n-1}$; this expression for d enables us to resolve the following problem, viz. *to insert between two quantities,* b *and* c, m *arithmetical means,* that is to say, quantities, which comprised between b and c, will form with them a progression by difference.

To resolve this problem, it will be sufficient to determine the ratio of the progression required. For this we have given the first term b, the last term c, and the number of terms $m+2$. Substituting therefore c, b, and $m+2$ for l, a, and n in the above expression for d, viz. $d = \frac{l-a}{n-1}$, the ratio required will be $\frac{c-b}{m+2-1} = \frac{c-b}{m+1}$, that is, to find the ratio sought, *we divide the difference of the two numbers* b *and* c *by the number of terms to be inserted plus* 1.

Let it be required, for example, to insert 11 arithmetical means between 17 and 77.

Here $$d = \frac{77-17}{12} = 5.$$

The progression required will therefore be

$$\div\, 17 \,.\, 22 \,.\, 27 \,.\, 32 \,.\,.\,.\,.\,.\, 72 \,.\, 77.$$

Let it be required, as a second example, to insert 9 arithmetical means between each antecedent and consequent of the progression $\div\, 2 \,.\, 5 \,.\, 8 \,.\, 11 \,.\, 14 \,.\,.\,.\,.$

It will readily be inferred from what has been done, that, *if between the terms of a progression by difference, taken two and two, we insert the same number of arithmetical means, the terms of this progression together with the arithmetical means inserted will form a progression by difference.*

QUESTIONS INVOLVING PROGRESSIONS BY DIFFERENCE.

1. A number consisting of 3 digits, which are in arithmetical progression, being divided by the sum of its digits gives a quotient 48; and if 198 be subtracted from it the digits will be inverted. Required the number.

Let $x =$ the second digit and y the common difference, the three digits will then be expressed by $x + y$, x, $x - y$.

Resolving the question we obtain $x = 3$; and the number required is 432.

2. Four numbers are in arithmetical progression. The sum of their squares is equal to 276, and the sum of the numbers themselves is equal to 32. What are the numbers?

Let $2y =$ the common difference and let $x + 3y$, $x + y$, $x - y$, $x - 3y$ be the numbers.

Resolving the question, we obtain for the numbers sought, 11, 9, 7 and 5.

3. A traveller sets out for a certain place and travels 1 mile the first day, 2 the second and so on. In 5 days afterwards another sets out and travels 12 miles a day. How long and how far must he travel to overtake the first?

Let $x =$ the number of days; then $x + 5 =$ the number of days the first travels, and $(x + 6)\dfrac{x + 5}{2} =$ the distance he travels.

Resolving the question, we obtain $x = 3$, or 10.

4. There are three numbers in arithmetical progression, whose sum is 21; and the sum of the first and second is to the sum of the second and third as 3 to 4. Required the numbers. Ans. 5, 7, 9.

5. From two towns, which were 168 miles distant, two persons, A and B, set out to meet each other; A went 3 miles the first day, 5 the next, and so on; B went 4 miles the first day, 6 the next, and so on. In how many days did they meet?

Ans. 8.

6. There are four numbers in arithmetical progression, whose sum is 28, and their continued product is 585. Required the numbers. Ans. 1, 5, 9, 13.

7. A and B, 165 miles distant from each other, set out with a design to meet; A travels 1 mile the first day, 2 the second, and so on; B travels 20 miles the first day, 18 the second, and so on. How soon will they meet?

Ans. They will meet in 10, and also in 33 days.

8. The sum of the squares of the extremes of four numbers in arithmetical progression is 200, and the sum of the squares of the means is 136. What are the numbers?

Ans. 14, 10, 6, 2, or $-14, -10, -6, -2$.

9. A regiment of men was just sufficient to form an equilateral wedge. It was afterwards doubled, but was still found to want 385 men to complete a square containing 5 more men in a side, than in a side of the wedge. How many did the regiment at first contain? Ans. 820.

PROGRESSION BY QUOTIENT.

174. A series of quantities such, that if any term be divided by the one which precedes it, the quotient is the same in whatever part of the series the two terms are taken, is called *a geometrical progression or progression by quotient.*

The constant quotient is called the *ratio* of the progression.

If the series is increasing, the ratio will be greater than unity; if decreasing, the ratio will be less than unity.

The following series are examples of this kind of progression,

$$3\,.\,6\,.\,12\,.\,24\,.\,48\,.\,96$$
$$64\,.\,16\,.\,4\,.\,1\,.\,\tfrac{1}{4}\,.\,\tfrac{1}{16}$$

In the first the ratio is 2, in the second $\frac{1}{4}$. A progression by quotient, it will readily be perceived, is simply a series of equal ratios by quotient, in which *each term is at once antecedent and consequent, with the exception of the first, which is only an antecedent, and of the last, which is only a consequent.*

To indicate that the quantities $a, b, c, d \ldots$ form a progression by quotient, we write them thus

$$\div\!\!\div\, a : b : c : d : \ldots$$

The progression is enunciated thus, a to b as b to c as c, &c.

175. Let us take the general progression

$$\div\!\!\div\, a : b : c : d \ldots.$$

and let the ratio be represented by q; from the nature of the progression, we have, it is evident,

$$b = aq,\ c = bq = aq^2,\ d = cq = aq^3$$

from which it will be readily inferred, that *a term of any rank whatever is equal to the first term multiplied by the ratio raised to a power, the exponent of which is one less than the number, which marks the place of this term.*

Let L designate any term whatever of the progression, and let n represent the number of this term; from what has been said, we have

$$L = aq^{n-1}.$$

This is called the *general term* of the progression. By means of it we may find any term required, when the first term and the ratio are given.

Thus let it be required to find the 8th term of the progression $\div\!\!\div\, 2 : 6 : 18 \ldots$ in this case, we have

$$L = 2 \times 3^7 = 4374.$$

s

In like manner if it be required to find the 12th term of the progression $\div\!\!\div 64 : 16 : 4 : 1 : \frac{1}{4} \ldots$, we have

$$L = 64\left(\frac{1}{4}\right)^{11} = \frac{1}{65536}.$$

176 Resuming the general progression

$$\div\!\!\div a : b : c : d \ldots k : l,$$

we have from the nature of the progression the series of equations,

$$b = aq,\ c = bq,\ d = cq \ldots l = kq;$$

adding these equations member to member, we have

$$b + c + d + \ldots l = (a + b + c + \ldots k)\,q. \quad (1)$$

Let S represent the sum of all the terms, we have

$$b + c + d + \ldots . l = S - a$$
$$a + b + c + \ldots . k = S - l;$$

whence by substitution in equation (1), we have

$$S - a = q\,(S - l),$$

and by consequence $$S = \frac{ql - a}{q - 1}.$$

By means of this formula we may obtain the sum of any number of the terms of a progression by quotient; for this purpose, *we multiply the last term by the ratio, subtract the first term from this product, and divide the remainder by the ratio diminished by unity.*

Let it be required to find the sum of the first 4 terms of the progression $\div\!\!\div 2 : 6 : 18 : 54 : 162$, we have

$$S = \frac{54 \times 3 - 2}{3 - 1} = 80.$$

When the progression is decreasing, that is, when q is less than 1, it will be more convenient to put the above expression for S under the form $S = \frac{a - lq}{1 - q}$; since in this case the two terms of the fraction will be positive.

Let it be required to find the sum of the 12 first terms of the progression $\div\!\!\div 64 : 16 : 4 : 1 : \frac{1}{4} \ldots . \frac{1}{65536}$.

We have $S=\frac{a-lq}{1-q}=\frac{64-\frac{1}{65536}\cdot\frac{1}{4}}{1-\frac{1}{4}}=85+\frac{65535}{196608}.$

177. If in the formulas for S we substitute for l its value, viz. $l=aq^{n-1}$, we have

$$S=\frac{aq^n-a}{q-1},\quad S=\frac{a-aq^n}{1-q},$$

formulas, by means of which we obtain the sum of any number of terms of a progression, when the number of terms, the first term and ratio are given.

Thus to find the sum of the first 8 terms of the progression $\div\!\!\div 2:6:18:54\ldots.$ we have

$$S=\frac{aq^n-a}{q-1}=\frac{2\times3^8-2}{3-1}=6560.$$

In the same manner, we have for the sum of the 12 first terms of the progression $\div\!\!\div 64:16:4\ldots$

$$S=\frac{a-aq^n}{1-q}=\frac{64-64(\frac{1}{4})^{12}}{1-\frac{1}{4}}=85+\frac{65535}{1966081}.$$

EXAMPLES.

1. The first term of a progression by quotient is 4, the ratio 3, and the last term 78372. What is the sum of all the terms?

2. The first term of a progression by quotient is 8, the last term $\frac{1}{2048}$, and the ratio $\frac{1}{2}$. What is the sum of the progression?

3. The first term of a progression by quotient is 3, the ratio $\frac{7}{5}$, and the number of terms 10. Required the sum of the progression.

4. The first term of a progression by quotient is $\frac{5}{6}$, the ratio $\frac{2}{3}$, and the number of terms 9. What is the sum of the progression?

INFINITE PROGRESSIONS BY QUOTIENT.

178. Let there be the decreasing progression

$$\div\!\!\div a : b : c : d \ldots$$

consisting of an infinite number of terms. The formula for the sum of any number of terms, viz. $S = \frac{a - aq^n}{1 - q}$ may be put under the form

$$S = \frac{a}{1-q} - \frac{a}{1-q} \cdot q^n.$$

But since the progression is decreasing, q is a fraction; q^n is also a fraction; hence as the number n becomes greater, or as we take more terms, the expression $\frac{a}{1-q} \cdot q^n$ becomes smaller, and the value of S approaches nearer to $\frac{a}{1-q}$. If then we suppose n greater than any assignable quantity or *infinite*, $\frac{a}{1-q} \cdot q^n$ will be less than any assignable quantity or 0, and $\frac{a}{1-q}$ will in this case represent the true value of the series.

We conclude, therefore, that the sum of the terms of a decreasing progression, in which the number of terms is infinite, has for its expression $S = \frac{a}{1-q}$, q being the ratio of the progression and a the first term.

179. Strictly speaking the quantity $\frac{a}{1-q}$ is the *limit* which the sum of a decreasing progression can never surpass, but to which it continually approximates as we take more terms.

Let there be, for example, the progression

$$\div\!\!\div 1 : \tfrac{1}{2} : \tfrac{1}{4} : \tfrac{1}{8} \ldots,$$

we have $\qquad a = 1,\ q = \frac{1}{2}$, whence

$$S = \frac{a}{1-q} - \frac{a}{1-q} \cdot q^n = \frac{2}{2-1} - \frac{2}{2-1} \times \left(\frac{1}{2}\right)^n = \frac{2}{2-1} - \frac{1}{2^{n-1}}.$$

Here the greater the value of n or the more terms we take, the less is the fraction $\frac{1}{2^{n-1}}$, and the nearer the sum of the series

approaches to 2. If the number of terms be considered infinite, the fraction $\frac{1}{2^{n-1}}$ will be less than any assignable quantity or zero, and the sum of the series will be equal to 2.

Strictly speaking, however, 2 is the limit, which the sum of the proposed series can never surpass, but to which it constantly approximates as we take more terms.

Thus let the number of terms be 1, 2, 3, 4 . . . . successively, we have

$$\begin{aligned}
1 &= 2-1\\
1+\tfrac{1}{2} &= 2-\tfrac{1}{2}\\
1+\tfrac{1}{2}+\tfrac{1}{4} &= 2-\tfrac{1}{4}\\
1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8} &= 2-\tfrac{1}{8}\\
1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16} &= 2-\tfrac{1}{16}.
\end{aligned}$$

Here the more terms we take, the nearer the sum of the progression will approach to 2, from which it may be made to differ by a quantity as small as we please, though strictly speaking, it can never become equal to 2.

180. When the series is increasing, that is, when q is greater than unity, the expression $S=\frac{a}{1-q}$ cannot be considered as the limit, which the sum of the series can never surpass. For the sum of a determinate number n of terms being $S=\frac{a}{1-q}-\frac{aq^n}{1-q}$, it is evident, that $\frac{aq^n}{1-q}$ will increase more and more numerically in proportion as n increases; by consequence the more terms we take, the more will the sum of the terms differ numerically from $\frac{a}{1-q}$. In this case $\frac{a}{1-q}$ is merely the algebraic expression, which by its development gives rise to the series

$$a+aq+aq^2+aq^3+\ldots.$$

Indeed if we perform upon a the division indicated, we have

$$\frac{a}{1-q}=a+aq+aq^2+aq^3+\ldots.$$

181. In the above expression let $a=1$, $q=2$, we have

$$\frac{1}{1-2} \text{ or } -1=1+2+4+8+16+\ldots$$

an equation, in which the first member is negative, while the second is positive, and greater in proportion as q is greater.

In order to interpret this result we observe, that if in the equation $\frac{a}{1-q}=a+aq+aq^2\ldots.$ we stop the series at any particular term, it is necessary, in order to preserve the equality of the two members, to complete the quotient by annexing to it the fraction which remains. If, for example, we stop the series at the fourth term aq^3, we shall have by completing the quotient

$$\frac{a}{1-q}=a+aq+aq^2+aq^3+\frac{aq^4}{1-q};$$

an equation which is exact. Indeed if in this equation, we make $a=1$, $q=2$, we have

$$-1=1+2+4+8+\frac{16}{-1};$$

from which we obtain $-1=-1$.

EXAMPLES.

1. What is the sum of the infinite progression

$$1:\tfrac{1}{3}:\tfrac{1}{9}:\tfrac{1}{27}:\ldots.$$

2. What is the sum of the progression $2:\frac{3}{2}:\frac{9}{8}:\ldots..$ continued to infinity?

3. The first term of a geometrical progression is a, and the ratio $-\frac{b}{a}$. What is the sum of this progression continued to infinity?

4. What vulgar fraction is equivalent to the repeating decimal 3?

This decimal may be put under the form

$$3\left(\tfrac{1}{10}+\tfrac{1}{100}+\tfrac{1}{1000}+\tfrac{1}{10000}+\ldots.\right)$$

5. What vulgar fraction is equal to the repeating decimal $\dot{2}\dot{5}$; what to the decimal $\dot{3}7\dot{5}$?

182. The equations $l=aq^{n-1}$, $S=\frac{lq-a}{q-1}$ contain all the

relations of the five quantities a, l, q, n, and S; we have then the general problem, *any three of the five quantities*, a, l, q, n *and* S *being given to find the remaining two.* This general problem gives rise to ten particular problems, the enunciations of which will not differ from those relative to progressions by difference, art. 171, with the exception, that the ratio is here expressed by the letter q instead of d.

183. From the formula $l = aq^{n-1}$, we obtain

$$q = \sqrt[n-1]{\frac{l}{a}}.$$

This expression for q enables us to resolve the following problem, viz. *to insert between two given numbers,* b *and* c, m *mean proportionals,* that is to say, a number m of *quantities, which comprised between* b *and* c *will form with them a progression by quotient.*

To resolve this problem it will be sufficient to determine the ratio of the progression required; for this we have given the first term b, the last term c and the number of terms $m+2$.

Substituting therefore b, c and $m+2$ for a, l and n in the above expression for q, we have for the ratio of the required progression

$$q = \sqrt[m+1]{\frac{c}{b}};$$

whence to find the ratio sought, *we divide the given numbers* b *and* c, *one by the other, and extract the root of the quotient to the degree marked by the number of terms to be inserted plus one.*

Let it be required to insert six *mean* proportionals between the numbers 3 and 384. Here $m = 6$, we have therefore

$$q = \sqrt[7]{\frac{384}{3}} = \sqrt[7]{128} = 2.$$

The progression required is therefore

$$\div\!\!\div 3 : 6 : 12 : 24 : 48 : 96 \quad 192 : 384.$$

From what has been done, it will be easy to see, that *if between the terms of a progression by quotient taken two and two, we in-*

sert the same number of mean proportionals, the partial progressions thus formed will together form a progression by quotient.

184. Of the ten particular problems, which may be proposed upon progressions by quotient, four only can be resolved by principles thus far laid down. Below we have the enunciation of these problems with their answers.

1°. a, q, n being given to find l and S.

$$l = aq^{n-1}, \quad S = \frac{a(q^n - 1)}{q - 1}.$$

2°. a, n, l being given to find q and S.

$$q = \sqrt[n-1]{\frac{l}{a}}, \quad S = \frac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}.$$

3°. q, n, l being given to find a and S.

$$a = \frac{l}{q^{n-1}}, \quad S = \frac{l(q^n - 1)}{q^{n-1}(q - 1)}.$$

4°. q, n, S being given to find a and l.

$$a = \frac{S(q - 1)}{q^n - 1}, \quad l = \frac{Sq^{n-1}(q - 1)}{q^n - 1}.$$

Of the remaining problems, two, viz. those in which a and q, l and q are the unknown quantities, depend upon the resolution of equations of a degree superior to the second. The other four depend upon the resolution of an equation of a nature altogether different from any which we have yet seen, viz. upon an equation of the form $a^x = b$ in which the exponent is the unknown quantity.

QUESTIONS PRODUCING PROGRESSIONS BY QUOTIENT.

1. There are three numbers in geometrical progression, the greatest of which exceeds the least by 15. Also the difference of the squares of the greatest and least is to the sum of the squares of all the three numbers as 5 to 7. Required the numbers.

Let x, xy, xy^2 be the numbers; then by the question we have

$$xy^2 - x = 15,$$

and

$$7(x^2y^4 - x^2) = 5(x^2y^4 + x^2y^2 + x^2),$$

or by division

$$7(y^4 - 1) = 5(y^4 + y^2 + 1).$$

or performing the operations indicated, transposing and reducing

$$y^4 - \frac{5}{2}y^2 = 6;$$

whence, resolving this last, we have

$$y^2 = 4, \text{ and } y = 2.$$

Substituting next for y its value in the first equation, we obtain $x = 5$. The numbers required are therefore 5, 10 and 20.

2. The sum of three numbers in geometrical progression is 13, and the product of the mean, and the sum of the extremes is 30. Required the numbers.

Let the numbers be $\frac{x}{y}$, x and xy; then by the question, we have

$$\frac{x}{y} + x + xy = 13,$$

and

$$\left(\frac{x}{y} + xy\right)x = 30.$$

By transposition in the first equation, we have

$$\frac{x}{y} + xy = 13 - x;$$

whence, by substitution in the second, we obtain

$$(13 - x)\,x = 30;$$

whence

$$x^2 - 13x = -30,$$

from which we deduce

$$x = 10,\ x = 3.$$

Substituting the value $x = 3$ in the first equation, we obtain $y = 3$, or $\frac{1}{3}$, and the numbers sought are 1, 3, 9.

3. The difference between the first and second of four numbers in geometrical progression is 36, and the difference between the third and fourth is 4. What are the numbers?

Ans. 54, 18, 6, and 2.

4. A gentleman divided £210 among three servants in geometrical progression; the first had £90 more than the last. How much had each?

5. There are three numbers in geometrical progression, the

sum of the first and second of which is 9, and the sum of the first and third is 15. Required the numbers.

6. The sum of three numbers in geometrical progression is 35, and the mean term is to the difference of the extremes as 2 to 3. Required the numbers. Ans. 5, 10, 20.

7. The sum of £14 was divided between three persons, whose shares were in geometrical progression; the sum of the shares of the first and second was to the sum of the shares of the second and third as 1 to 2. Required the shares.

Ans. 2, 4, 8.

SECTION XXII.—Theory of Continued Fractions.

185. In order to form a more exact idea of a fraction, the terms of which are large numbers and prime to each other, we seek approximate values of this fraction, which are expressed in more simple numbers.

Let there be, for example, the fraction $\frac{159}{493}$. Dividing both terms of this fraction by the numerator, an operation which will not change its value, it becomes $\cfrac{1}{3+\cfrac{16}{159}}$.

If then we neglect, for the moment, the fraction $\frac{16}{159}$ in this expression, the result $\frac{1}{3}$ will be greater than the proposed, since the denominator has been diminished.

On the other hand, if instead of neglecting the fraction $\frac{16}{159}$, we substitute 1 for it, the result $\frac{1}{4}$ will be less than the proposed, since the denominator has been increased.

We conclude therefore, that the fraction $\frac{159}{493}$ is comprised between $\frac{1}{3}$ and $\frac{1}{4}$, we are thus enabled to form a very exact idea of its value.

If a greater degree of approximation be required, we have only to operate upon $\frac{16}{159}$ in the same manner as we have already done upon $\frac{159}{493}$; we have thus

$$\frac{16}{159}=\cfrac{1}{9+\frac{15}{16}}$$

and the proposed fraction becomes

$$\cfrac{1}{3+\cfrac{1}{9+\frac{15}{16}}}.$$

If we neglect $\frac{15}{16}$, $\frac{1}{9}$ is greater than $\frac{16}{159}$; it follows therefore that $\cfrac{1}{3+\frac{1}{9}}$ is less than $\frac{159}{493}$; but $\cfrac{1}{3+\frac{1}{9}}$ becomes $\cfrac{1}{\frac{28}{9}}$ or $\frac{9}{28}$; thus the proposed is comprised between $\frac{1}{3}$ and $\frac{9}{28}$. The difference between these two fractions is $\frac{1}{84}$; the error therefore committed in taking $\frac{1}{3}$ or $\frac{9}{28}$ for the value of the proposed fraction is less than $\frac{1}{84}$.

To attain to a still greater degree of approximation, we operate in the same manner upon $\frac{15}{16}$; thus we have

$$\frac{15}{16}=\cfrac{1}{1+\frac{1}{15}}$$

and the proposed fraction becomes

$$\cfrac{1}{3+\cfrac{1}{9+\cfrac{1}{1+\frac{1}{15}}}}.$$

Neglecting $\frac{1}{15}$, the fraction $\frac{1}{1}$ or 1 is greater than $\frac{15}{16}$; hence $\cfrac{1}{9+\cfrac{1}{1}}$ or $\frac{1}{10}$ is less than $\frac{16}{159}$; therefore $\cfrac{1}{3+\cfrac{1}{9+\cfrac{1}{1}}}$ or $\frac{10}{31}$ is greater than $\frac{159}{493}$; thus the proposed is comprised between $\frac{9}{28}$ and $\frac{10}{31}$. The difference between these two fractions is $\frac{1}{868}$; the error committed, therefore, in taking either $\frac{9}{28}$ or $\frac{10}{31}$ for the value of the proposed is less than $\frac{1}{868}$.

The expression $\cfrac{1}{3+\cfrac{1}{9+\cfrac{1}{1+\cfrac{1}{15}}}}$ is called a *continued fraction.* We understand therefore, by a continued fraction *a fraction, which has unity for its numerator, and for its denominator an entire number plus a fraction, which fraction has also unity for its numerator and for its denominator an entire number plus a fraction, and thus in order.*

It sometimes happens, that the proposed fractional number is greater than unity; to generalize, therefore, the above definition, we say, that a continued fraction is *an expression composed of an entire number plus a fraction which has unity for its numerator, and for its denominator an entire number plus a fraction, &c.*

186. If we examine the above process for converting $\frac{159}{493}$ into a continued fraction, it will be perceived, that we have divided first 493 by 159, which gives three for a quotient and a remainder 16; we then divide 159 by 16, which gives 9 for a quotient and a remainder 15; we next divide 16 by 15, which gives 1 for a quotient and a remainder 1; from which we readily infer the following rule for converting a fraction or fractional number into a continued fraction, viz.

Apply to the two terms of the fraction the process of finding their greatest common divisor; pursue the operation until a remainder is obtained equal to 0; *the successive quotients, thus obtained, will be the denominators of the fractions, which constitute the continued fraction.*

If the proposed be greater than unity the first quotient will be the entire part, which enters into the expression of the continued fraction.

EXAMPLES. Let the fractions $\frac{73}{137}$, $\frac{829}{347}$ be converted into continued fractions.

187. From what has been said a continued fraction may be represented generally by the expression

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\ldots}}}$$

$a, b, c, d \ldots$ being entire and positive numbers. The fractional number, to which this expression is equivalent, may moreover be represented by x.

The fractions $\frac{1}{b}, \frac{1}{c}, \frac{1}{d} \ldots$, the assemblage of which constitutes the continued fraction, are called *integrant fractions.* The denominators $b, c, d \ldots$ are called *incomplete quotients,* since b, for example, is only the entire part of the number expressed by $b+\cfrac{1}{c+\cfrac{1}{d+\ldots}}$ and c only the entire part of the number expressed by $c+\cfrac{1}{d+\ldots}$ and thus in order. Conversely the expressions $b+\cfrac{1}{c+\cfrac{1}{d\ldots}}$ $c+\cfrac{1}{d\ldots}$ are called *complete quotients.*

The results obtained by converting successively into a single fractional number each of the expressions

$$a+\frac{1}{b},\ a+\cfrac{1}{b+\cfrac{1}{c}}\ \text{\&c. are called } \textit{reductions.}$$

188. The formation of these reductions is subject to a very simple law, which we shall now develop.

The first is a, which may be put under the form $\frac{a}{1}$, the second is $a+\frac{1}{b}$, or reducing the whole expression to a fraction, $\frac{ab+1}{b}$. To form the third, represented by

$$a+\cfrac{1}{b+\cfrac{1}{c}},$$

it will be sufficient to substitute $b+\frac{1}{c}$ for b in the second; making this substitution, we have

$$a+\cfrac{1}{b+\cfrac{1}{c}}=\frac{a\left(b+\frac{1}{c}\right)+1}{b+\frac{1}{c}}=\frac{(ab+1)c+a}{bc+1}.$$

To form the fourth reduction, it will be sufficient to substitute $c+\frac{1}{d}$ for c in the third; which gives

$$\frac{[(ab+1)c+a]d+ab+1}{(bc+1)d+b}$$

The first four reductions therefore will be

$$\frac{a}{1},\quad \frac{ab+1}{b},\quad \frac{(ab+1)c+a}{bc+1},\quad \frac{[(ab+1)c+a]d+ab+1}{(bc+1)d+b}.$$

Without, proceeding further, it will be perceived, that the numerator of the third reduction is formed by multiplying the numerator of the second by the third incomplete quotient c, and adding to this product the numerator of the first reduction. With respect to the denominator, it is formed in the same manner by means of the denominators of the second and first reductions.

In like manner, the numerator and denominator of the fourth reduction is formed, it will be perceived, by multiplying respectively the two terms of the third reduction by the fourth

incomplete quotient d and adding to the two products respectively the two terms of the second reduction.

From what has been done it will be readily inferred, that the above law of formation for the third and fourth reductions should be extended to those which follow. To demonstrate this law, however, in a rigorous manner, we shall show that if it be true in regard to any three successive reductions whatever, it will be true for the reduction, which follows; thus this law being already found true for the first three reductions will be true for the fourth, and being true for the second, third and fourth, it will be true for the fifth, and thus in order; it will therefore be general.

Let $\frac{P}{P'}$, $\frac{Q}{Q'}$, $\frac{R}{R'}$ be any three successive reductions whatever; let r be the incomplete quotient, at which we stop in order to form the reduction $\frac{R}{R'}$, and let it be supposed, that we have

$$\frac{R}{R'}=\frac{Qr+P}{Q'r+P'}.$$

Let $\frac{1}{s}$ be the integrant fraction, which follows r, and let $\frac{S}{S'}$ be the corresponding reduction. In order to form this reduction, it is sufficient to substitute in the expression for $\frac{R}{R'}$, $r+\frac{1}{s}$ instead of r; making this substitution, we have

$$\frac{S}{S'}=\frac{Q\left(r+\frac{1}{s}\right)+P}{Q'\left(r+\frac{1}{s}\right)+P'}=\frac{(Qr+P)s+Q}{(Q'r+P')s+Q'}=\frac{Rs+Q}{R's+Q'}.$$

We see, therefore, that $\frac{S}{S'}$ is formed from the two preceding reductions according to the law enunciated above. This law is therefore general; whence, *To form the numerator of any reduction whatever, we multiply the numerator of the preceding reduction by the incomplete quotient, which corresponds to it, and add to the product the numerator of the reduction, which precedes by two ranks the one which we wish to form; the denominator is*

formed by the same law by means of the two preceding denominators.

189. When the number reduced to a continued fraction is less than unity, we substitute $\frac{0}{1}$ instead of a, in order to apply the law, which supposes necessarily, that we have already the first two reductions.

Let it be proposed to find the successive reductions of the continued fraction

$$\frac{65}{149}=\frac{0}{1}+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\frac{1}{2}}}}}}$$

The first two reductions being $\frac{0}{1}$, $\frac{1}{2}$, we have for those which follow

$$\frac{3}{7},\ \frac{7}{16},\ \frac{17}{39},\ \frac{24}{55},\ \frac{65}{149}.$$

In like manner we have for the several reductions of the continued fraction arising from $\frac{829}{347}$,

$$\frac{2}{1},\ \frac{5}{2},\ \frac{7}{3},\ \frac{12}{5},\ \frac{43}{18},\ \frac{829}{347}.$$

So also the fraction $\frac{29}{77}$ being converted into a continued fraction gives the following reductions, viz.

$$\frac{0}{1},\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{5},\ \frac{3}{8},\ \frac{29}{77}.$$

190. The successive reductions, it will be perceived, are alternately less and greater than the whole continued fraction, and they approximate this fraction nearer and nearer.

The first reduction is always less than the whole continued fraction x. *The reductions of an even rank are, therefore, greater than the whole continued fraction, and those of an odd rank are less.* And since these reductions approach nearer and nearer

the value of x, the reductions of an odd rank must go on increasing, while those of an even rank decrease. Thus the reductions form two *series*, the terms of which approach nearer and nearer the value of the whole continued fraction.

191. The difference between any two consecutive reductions whatever has unity for its numerator. The numerator of the first difference is $+1$, that of the second -1, that of the third $+1$, and thus in order. In general, *the numerator of any difference whatever will be* $+1$, *if the second of the reductions under consideration is of an even rank, but* -1 *if it be of an odd rank.*

From this property, it follows, that the two terms of any reduction whatever $\frac{R}{R'}$ are prime to each other.

Indeed let it be supposed, that R and R' have a common factor h; by the preceding property, we have

$$RQ' - QR' = \pm 1;$$

whence dividing both terms by h, we have

$$\frac{RQ'}{h} - \frac{QR'}{h} = \frac{1}{h};$$

but the first member of this equation is an entire number since by hypothesis R and R' are divisible by h, while the second is essentially a fraction; R and R' cannot therefore have a common factor.

From this it follows, that if a fraction, the terms of which are not prime to each other, be converted into a continued fraction, and all the reductions be formed to the last inclusive, the last reduction will not be the proposed fraction, but this fraction reduced to its lowest terms.

Let there be, for example, the fraction $\frac{348}{924}$; converting this into a continued fraction, we have for the successive reductions $\frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{8}, \frac{29}{77}$. The last reduction $\frac{29}{77}$ is the proposed reduced to its lowest terms.

192. Since the value of the whole continued fraction x is

always comprised between any two consecutive reductions $\frac{Q}{Q'}$, $\frac{R}{R'}$, it follows that the error committed in taking $\frac{Q}{Q'}$ for x is less than $\frac{Q}{Q'}-\frac{R}{R'}$; but from what has been said we have

$$\frac{Q}{Q'}-\frac{R}{R'}=\frac{1}{Q'R'}$$

and since $Q'<R'$ gives $Q'^2<Q'R'$, we have

$$\frac{1}{Q'R'}<\frac{1}{Q'^2}.$$

The error therefore committed in taking any reduction whatever for the value of the whole continued fraction *is less than unity divided by the denominator of this reduction multiplied by the denominator of the reduction which follows*, or less exactly but in terms more simple, *less than unity divided by the square of the denominator of the reduction, which is taken for the whole continued fraction.*

193. The ratio of the circumference to the diameter of a circle being expressed by the fraction $\frac{314159}{100000}$, the terms of which are prime to each other, let it be proposed to find a fraction, the terms of which will be more simple, and which will express the same ratio nearly. Converting the proposed into a continued fraction, we have for the successive reductions

$$\frac{3}{1},\ \frac{22}{7},\ \frac{333}{106},\ \frac{355}{113},\ \frac{9208}{2931},\ \frac{9563}{3044},\ \frac{76149}{24239},\ \frac{314159}{100000}.$$

The error committed in taking the second of these reductions for the proposed fraction will not exceed $\frac{1}{742}$; $\frac{22}{7}$ is therefore frequently employed to express the ratio of the circumference of a circle to its diameter. This is the ratio given by *Archimedes.*

If a greater degree of approximation is required, we take the fourth reduction, which it is easy to see, is but little more complicated than the third. The error committed in taking this reduction for the proposed will not exceed $\frac{1}{113\times 2931}$; $\frac{355}{113}$ will

therefore approximate the proposed very nearly. This is the ratio given by *Adrian Metius.*

We thus see the use, which may be made of continued fractions in estimating approximatively the value of fractions, the terms of which are large numbers and prime to each other.

EXAMPLES.

1. A's property is to that of B as 5743 to 80937. By what smaller numbers may the ratio of their property be expressed?

2. The lunar month or the time in which the moon completes its revolution, is found by calculation to be 27.321661 days. Thus in 27321661 days it would perform 1000000 revolutions. How may this relation be expressed in smaller numbers?

SECTION XXIII.—Exponential Equations and Logarithms.

194. An equation of the form $a^x = b$, in which the exponent x is the unknown quantity is called an *exponential equation.* The solution of this equation consists in finding the power, to which it is necessary to raise a given quantity a in order to produce another given quantity b.

Let there be, for example, the equation $2^x = 64$; raising 2 to its different powers, we soon find that $2^6 = 64$; $x = 6$ answers therefore the conditions of the equation.

Again, let there be the equation $3^x = 243$; raising 3 to its different powers we find $3^5 = 243$; whence $x = 5$. In a word, so long as the second member b is a perfect power of the given number a, x will be an entire number and its value may be found by raising a successively to its different powers, beginning with that, the exponent of which is 0.

Let it be proposed next to resolve the equation $2^x = 6$. Putting successively $x = 2$, $x = 3$, we have $2^2 = 4$, $2^3 = 8$; the value of x is, therefore, comprised between the numbers 2 and 3.

Let us put therefore, $x=2+\frac{1}{x'}$, x' being greater than 1, substituting this value in the proposed, we have

$$2^{2+\frac{1}{x'}}=6 \text{ or } 2^2\times 2^{\frac{1}{x'}}=6, \text{ whence } 2^{\frac{1}{x'}}=\frac{3}{2},$$

or raising both members to the power x', we have

$$\left(\frac{3}{2}\right)^{x'}=2.$$

To determine the value of x' we make successively $x'=1$, $x'=2$; thus we have $\left(\frac{3}{2}\right)$ or $\frac{3}{2}$ less than 2, but $\left(\frac{3}{2}\right)^2$ or $\frac{9}{4}$ greater than 2; x' is, therefore, comprised between 1 and 2.

Let us put then $x'=1+\frac{1}{x''}$, x'' being greater than 1. Substituting this value, we have

$$\left(\frac{3}{2}\right)^{1+\frac{1}{x''}}=2 \text{ or } \frac{3}{2}\times\left(\frac{3}{2}\right)^{\frac{1}{x''}}=2;$$

whence $$\left(\frac{4}{3}\right)^{x''}=\frac{3}{2}.$$

To determine the value of x'', we make successively $x''=1$, $x''=2$; thus we have $\left(\frac{4}{3}\right)$ or $\frac{4}{3}$ less than $\frac{3}{2}$, but $\left(\frac{4}{3}\right)^2$ or $\frac{16}{9}$ greater than $\frac{3}{2}$; x'' is, therefore, comprised between 1 and 2.

Let us put then $x''=1+\frac{1}{x'''}$, x''' being greater than unity; we have by substitution

$$\left(\frac{4}{3}\right)^{1+\frac{1}{x'''}}=\frac{3}{2} \text{ or } \frac{4}{3}\times\left(\frac{4}{3}\right)^{\frac{1}{x'''}}=\frac{3}{2}; \text{ whence } \left(\frac{9}{8}\right)^{x'''}=\frac{4}{3}.$$

Making successively $x'''=1, 2, 3$, we find $\left(\frac{9}{8}\right)^2=\frac{81}{64}$, a number less than $\frac{4}{3}$ but $\left(\frac{9}{8}\right)^3=\frac{729}{512}$, a number greater than $\frac{4}{3}$; thus x''' is comprised between 2 and 3.

Let $x\;\;=2+\frac{1}{x''''}$, the equation in x''' becomes

$$\left(\frac{9}{8}\right)^{2+\frac{1}{x''''}}=\frac{4}{3}; \text{ whence } \left(\frac{256}{243}\right)^{x''''}=\frac{9}{8}.$$

Operating upon this last equation as upon the preceding, we find two entire numbers k and $k+1$, between which x'''' will be comprised. Putting $x''''=k+\frac{1}{x^{v}}$, we determine x^{v}, in the same manner as we have already done x'''', and thus in order.

Bringing together the equations

$$x=2+\frac{1}{x'},\quad x'=1+\frac{1}{x''},\quad x''=1+\frac{1}{x'''},\quad x'''=2+\frac{1}{x''''}\,..$$

we obtain the value of x under the form of a continued fraction, thus

$$x=2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{x''''}}}}$$

But we have seen that in a continued fraction the greater the number of integrant fractions, which are taken, the nearer we approach the value of the number reduced to a continued fraction; we shall, therefore, be able to determine the value of x in the equation $2^x=6$, if not exactly, at least with such degree of approximation as we please.

Forming the first four reductions, for example, we have $\frac{2}{1}, \frac{3}{1}, \frac{5}{2}, \frac{13}{5}$; and the reduction $\frac{13}{5}$ differs from x by a quantity less than $\frac{1}{25}$.

To attain a greater degree of approximation, we determine the value of x'''' in the equation $\left(\frac{256}{243}\right)^{x''''}=\frac{9}{8}$; we thus find $x''''=2+\frac{1}{x^{v}}$. We shall have, then, for the fifth reduction $\frac{31}{12}$. This differs from x by a quantity less than $\frac{1}{144}$.

195. From the preceding examples the course to be pursued in the solution of equations of the form $a^x=b$ will be readily

inferred. In the application of this method to particular cases it is necessary to remark, 1°. If the quantity b be less than a, the value of x will be comprised between 0 and 1; we put, therefore, $x=\frac{1}{x'}$. 2°. If b is a fraction and a greater than unity, the value of x will be negative, we put, therefore, $x=-y$; the equation is then reduced to the form $a^y=\frac{1}{b}$; having found the value of y in this equation according to the method explained above, the value of x will be equal to that of y taken *negatively*.

EXAMPLES.

1°. Given $3^x=15$ to find the value of x. Ans. $x=2.46$.

2°. . $10^x=3$ . . . . . Ans. $x=0.47$.

3°. . $5^x=\frac{2}{3}$ to find the value of x. Ans. $x=-0.25$.

4°. . $\left(\frac{7}{12}\right)^x=\frac{3}{4}$ . . . . Ans. $x=0.53$.

In the above examples the reductions furnished by the method are converted into decimal fractions, and the value of x is determined to within .01.

THEORY OF LOGARITHMS.

196. If in the equation $a^x=y$, we assign a constant value different from unity to a, and suppose that of x to vary, as may be required, we may obtain successively for y all possible numbers.

Let us suppose first a greater than 1.

If we make successively $\quad x=0,\ 1,\ 2,\ 3,\ 4,\ \ldots.$

we have $\quad y=1,\ a,\ a^2,\ a^3,\ a^4,\ \ldots.$

Thus by means of the powers of a, the exponents of which are positive, entire or fractional, we may produce all possible positive numbers greater than 1.

Again, let $x=0,\ -1,\ -2,\ -3,\ -4,\ \ldots.$

we have $\quad y=1,\ \frac{1}{a},\ \frac{1}{a^2},\ \frac{1}{a^3},\ \frac{1}{a^4},\ \ldots.$

Thus by means of the powers of a, the exponents of which are negative entire or fractional, we may produce all possible positive numbers less than 1.

If on the other hand we suppose a less than unity, still all possible positive numbers may be produced by means of the different powers of a, only the order in which they are produced will be reversed.

We see therefore, *that all possible positive numbers may be produced by means of any positive number whatever* a, *different from unity, by raising this number to the requisite powers.*

It is necessary, that a should be different from unity, otherwise the same number will be produced, whatever value we assign to x.

197. Let it now be supposed that we have made a table containing in one column all entire numbers, and by the side of these in another column the exponents of the powers, to which it is necessary to raise a constant number in order to produce these numbers; this would be *a table of logarithms.*

The logarithm of a number, is, therefore, *the exponent of the power, to which it is necessary to raise a given or invariable number, in order to produce the proposed number.*

Thus in the equation $a^x = y$, x is the logarithm of y; in like manner in the equation $2^6 = 64$, 6 is the logarithm of 64. The logarithm of a number is indicated by writing before it the first three letters of the word logarithm, or more simply by placing before it the letter L.

The invariable number, from which the others are formed is called the *base* of the table. It may be taken at pleasure either greater or less than unity, but should remain the same for the formation of all numbers, that belong to the same table.

Since $a^0 = 1$, and $a^1 = a$, whatever number may be assumed for the base of the table, *the logarithm of the base will be unity and the logarithm of unity will be* 0.

198. We proceed to show the properties of logarithms in relation to numerical calculations.

1. Let there be the series of numbers $y, y', y'', \ldots.$ to be multiplied together. Let a represent the base of a system of logarithms, which we suppose already calculated, and let x, x', x'' . . be the logarithms of $y, y', y'', \ldots$; by the definition of a logarithm we have

$$y = a^x, \; y' = a^{x'}, \; y'' = a^{x''};$$

multiplying these equations member by member, we have

$$y y' y'' = a^{x+x'+x''},$$

whence $\log y y' y'' = x + x' + x'' = \log y + \log y' + \log y''$.

That is, ***the logarithm of a product is equal to the sum of the logarithms of the factors of this product.***

If then a multiplication be proposed, we take from a table of logarithms the logarithms of the numbers to be multiplied; the sum of these logarithms will be the logarithm of the product sought. Seeking therefore this logarithm in the table, the number corresponding to it will be the product sought. Thus by means of a table of logarithms ***addition may be made to take the place of multiplication.***

2. Let it be required to divide the number y by the number y'; let x, x' be the logarithms of these numbers, we have the equations

$$y = a^x, \; y' = a^{x'};$$

dividing these equations member by member, we have

$$\frac{y}{y'} = \frac{a^x}{a^{x'}} = a^{x-x'},$$

whence $$\log \frac{y}{y'} = x - x' = \log y - \log y'.$$

That is, ***the logarithm of a quotient is equal to the difference between the logarithm of the divisor and that of the dividend.***

If then it be proposed to divide one number by another, from the logarithm of the dividend we subtract the logarithm of the divisor, the result will be the logarithm of the quotient; seeking therefore this logarithm in the tables the number corresponding will be the quotient sought. Thus, by means of a table of logarithms, ***subtraction may be made to take the place of division.***

3. Let it next be required to raise the number y to the power denoted by m, we have the equation $\quad a^x = y;$
raising both members to the mth power, we have

$$a^{mx} = y^m;$$

whence the logarithm of $y^m = mx = m \log y$.

That is, *the logarithm of any power of a number is equal to the product of the logarithm of this number by the exponent of the power.*

To form any power whatever of a number by means of a table of logarithms, we multiply, therefore, the logarithm of the proposed number by the exponent of the power, to which it is to be raised; the number in the table corresponding to this product, will be the power sought.

4. Again, let it be required to find the nth root of y. We have as before $\quad a^x = y;$
whence taking the nth root of both members, we have

$$a^{\frac{x}{n}} = y^{\frac{1}{n}}; \text{ whence } \log y^{\frac{1}{n}} = \frac{x}{n} = \frac{\log y.}{n}$$

That is, *the logarithm of the root of any degree whatever of a number is equal to the logarithm of this number divided by the index of the root.*

Thus by the aid of a table of logarithms *a number may be raised to a power by a simple multiplication, and its root may be extracted by a simple division.*

FORMATION OF TABLES.

199. The properties of logarithms demonstrated above are altogether independent of the number a or their base. We may therefore form an infinite variety of tables of logarithms by putting for a all possible numbers except unity.

If it be required to construct a table of logarithms the base of which is 2, in the equation $2^x = y$, we make y equal successively to the numbers 1, 2, 3 . . . ., and determine by the methods explained, art. 195, the values of x corresponding.

We thus obtain the values of x exactly, if y be a perfect power of 2, or otherwise with such degree of approximation as we please

To calculate the logarithm of 3, for example, we have the equation $2^x = 3$, from which we deduce

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{x'''}}}}}$$

Whence stopping at the fourth integrant fraction, and forming the reduction corresponding, we have $x = \frac{19}{12}$, or reducing this last to a decimal we have $x = 1.583$ accurate to the third decimal figure.

200. In the calculation of a table of logarithms, it will be sufficient to calculate directly the logarithms of the prime numbers 1, 2, 3, 5 . . ., the logarithms of compound numbers may then be obtained by adding the logarithms of the prime factors, which enter into them. To find the logarithm of 35, for example, we have $35 = 5 \times 7$; whence $\log 35 = \log 5 + \log 7$; having already calculated the logarithms of 5 and 7, the logarithm of 35 will be found therefore by adding the logarithm of 5 to that of 7.

Since moreover the logarithm of a fraction will be equal to the logarithm of the numerator minus the logarithm of the denominator, it will be sufficient to place in the tables the logarithms of entire numbers.

201. Below we have a table of logarithms of numbers from 1 to 30 inclusive, the base of the system is 2, and the logarithms are calculated to 4 places of decimals.

N.	Log.	N.	Log.	N.	Log.
1	0.0000	11	3.4594	21	4.3922
2	1.0000	12	3.5849	22	4.4594
3	1.5849	13	3.7000	23	4.5235
4	2.0000	14	3.8073	24	4.5849
5	2.3219	15	3.9065	25	4.6438
6	2.5849	16	4.0000	26	4.7000
7	2.8073	17	4.0874	27	4.7548
8	3.0000	18	4.1699	28	4.8073
9	3.1699	19	4.2479	29	4.8577
10	3.3219	20	4.3219	30	4.9065

202. The most convenient number for a base to a system of logarithms, and the one employed in the construction of the tables in common use is 10.

If in the equation $10^x = y$ we make successively

$$x = 0, \quad 1, \quad 2, \quad 3, \quad 4 \ldots$$

we have $$y = 1, \quad 10, \quad 100, \quad 1000, \quad 10000 \ldots$$

Again if we make

$$x = 0, \; -1, \; -2, \; -3, \; -4 \ldots$$

we have $$y = 1, \quad \frac{1}{10}, \; \frac{1}{100}, \; \frac{1}{1000}, \; \frac{1}{10000}, \ldots$$

Therefore in a table of logarithms, the base of which is 10, 1°. the logarithms of numbers greater than unity are positive and go on increasing from 0 to infinity. 2°. The logarithms of numbers less than unity are negative, and their absolute values are so much the greater as the fractions are smaller; whence if we take a fraction less than any assignable quantity, the logarithm of this fraction will be negative, and its absolute value will be greater than any assignable quantity. On this account we say that the logarithm of 0 is an *infinite negative quantity.* 3°. The logarithms of all numbers below 10 are fractions; the logarithms of numbers between 10 and 100 are 1 and a fraction; the logarithms of numbers between 100 and 1000 are 2 and a fraction; those of numbers between 1000 and 10000 are 3 and a fraction; and in general, the whole number which precedes the fraction in the logarithm is less by one than the number of figures in the number corresponding to the logarithm. On this account it is called the *index or characteristic* of the logarithm, since it serves to indicate the order of units, to which the number corresponding to the logarithm belongs. Thus in the logarithm 3.75527 the characteristic 3 shows that the number corresponding to this logarithm consists of 4 figures or is comprised between 1000 and 10000.

203. The logarithm of a number being given, the logarithm of a number 10, 100, . . . times greater is found by adding 1, 2, . . . units to the characteristic only; indeed log

$$(y \times 10^n) = \log y + \log 10^n = \log y + n;$$

whence it will be sufficient to add n units to the logarithm of y in order to obtain the logarithm of a number 10^n times as great; an addition which may be performed upon the characteristic only. Conversely, $\log \frac{y}{10^n} = \log y - \log 10^n = \log y - n$; thus it is sufficient to subtract n units from the logarithm of y, in order to find the logarithm of a number 10^n times smaller than y.

204. The fractional parts of logarithms in the tables are expressed by decimals. From what has been said the decimal part of the logarithm of a number will be the same for this number multiplied or divided by 10, 100, . . . On this account the system of logarithms, the base of which is 10, is more convenient than any other system, since we have frequent occasion to multiply or divide by 10, 100, . . . operations reduced in this case to the simple addition or subtraction of units.

205. Since the characteristic of the logarithm may be easily determined by the number, and the number of figures in the number by the characteristic of the logarithm, it is usual to omit the characteristic in the tables to save the room. It is also convenient to omit it; because the same decimal part with different characteristics forms the logarithms of several different numbers.

206. Having already calculated a system of logarithms, it will be easy from this to form as many other systems as we please.

Indeed, let N designate any number whatever, log N its logarithm in the system the base of which is a, X its logarithm in a different system the base of which is b, we have

$$b^X = N.$$

Taking the logarithms of both members of this equation in the system, the base of which is a, we have

$$X . \log b = \log N;$$

whence
$$X = \frac{\log N}{\log b}.$$

Having calculated therefore a set of tables for a particular base, to find the logarithm of a number in a proposed system with a different base, *we take from the tables already calculated the logarithm of the number, and also the logarithm of the base of the proposed system; the former of these logarithms, divided by the latter, will give the logarithm of the number in the proposed system.*

The logarithm of 6, for example, in the system, the base of which is 10, is .77815, and that of 3 is .47712; the logarithm of 6, therefore, in the system the base of which is 3, will be

$$\frac{77815}{47712} = 1.63093.$$

207. The expression $X = \frac{\log N}{\log b}$ may be put under the form $X = \frac{1}{\log b} \log N$. Thus having already formed a table of logarithms, the base of which is a, to construct from this a new table, the base of which shall be b, we multiply the logarithms of the first table by the quantity $\frac{1}{\log b}$. This quantity by means of which we are enabled to pass from the old to the new table, is called the *modulus* of the new table in relation to the old.

MODE OF USING THE TABLES.

208. As it is impossible to place in the tables the logarithms of all numbers, it is usual to place in them the logarithms of numbers from unity to within a certain limit. In what follows it is supposed, that the student has in his hands tables containing the logarithms of entire numbers from 1 to 10000.

In order to use such a set of tables, we have the two following questions to resolve, viz. 1°. *Any number whatever being given, to find its logarithm.* 2°. *Any logarithm being given, to find the number which corresponds to it.*

The following examples will exhibit the method of resolving these questions.

1. Let it now be proposed to find the logarithm of 9748

Seeking the proposed in the column of numbers, against it in the column of logarithms we find 98892; this will be the decimal part of the logarithm; or, as is the case with most tables, if the column of numbers contain but three places of figures, we look for 974, the first three figures of the proposed, in the first column, and at the top of the table we look for the fourth figure 8; directly under the 8 and in the same line with 974, we find the decimal part 98892 as before; then since the proposed consists of four places, the characteristic will be 3, thus $\log 9748 = 3.98892$.

2. Let it be required to find the logarithm of 76.93. Removing for the moment the decimal point, we find as above $\log 7693 = 3.88610$, whence, art. 203, subtracting 2 units from the characteristic 3 of this logarithm, we shall have the logarithm of the proposed; thus $\log 76.93 = 1.88610$.

3. To find the logarithm of .75. The logarithm of this number may be presented under two different forms. Writing it in the form of a vulgar fraction, it becomes $\frac{75}{100}$. The logarithm of 75 is 1.87506, and that of 100 is 2.00000; whence subtracting the logarithm of the denominator from that of the numerator, art. 198, we have $-12494 = \log .75$. This logarithm, being altogether negative, is inconvenient in practice; it will be observed, however, that $.75 = \frac{1}{100} \times 75$; whence

$$\log .75 = \log \frac{1}{100} + \log 75 = -2 + 1.87506,$$
$$= -1 + 87506,$$

or placing the sign $-$ over the 1 to show that the characteristic only is negative, we have $\log .75 = \bar{1}.87506$.

This last form of the logarithm of the proposed is derived, it will be perceived, immediately from the continuation of the principle, art. 203, according to which the logarithm of a number 10, 100 . . . times less than a proposed number is found by subtracting 1, 2 . . . units from the characteristic of its logarithm.

Thus since the logarithm of 750 = 2.87506, we have

$$\log 75 = 1.87506$$
$$\log 7.5 = 0.87506$$
$$\log .75 = \bar{1}.87506$$
$$\log .075 = \bar{2}.87506$$
$$\log .0075 = \bar{3}.87506$$

4. To find the logarithm of $\frac{4}{5}$; we have $\log 4 = .60206$, $\log 5 = .69897$; whence subtracting this last logarithm from the former, we have $\log \frac{4}{5} = -.09691$, in which the logarithm is entirely negative. But $\frac{4}{5}$ reduced to a decimal becomes .8, the logarithm of which is $\bar{1}.90309$, the characteristic only being negative.

5. To find the logarithm of $54\frac{7}{9}$; we have $54\frac{7}{9} = \frac{493}{9}$; $\log 493 = 2.69285$, $\log 9 = 0.95424$; whence subtracting the latter logarithm from the former, we have $\log \frac{493}{9}$ or $54\frac{7}{9} = 1.73861$.

6. To find the logarithm of 675437. This number exceeds the limits of the table; its logarithm, however, may be readily found. The greatest number of places in a number, the logarithm of which can be found in the tables, is 4; separating therefore the four left hand figures of the proposed from the rest by a point, we consider for the moment those on the right as decimals. The logarithm of 6754.37 is comprised between the logarithm of 6754 and that of 6755; the difference between these two logarithms is .00007; $\frac{37}{100}$ of this difference therefore added to the less logarithm will give the logarithm of 6754.37 nearly; thus $\log 6754.37 = 3.82959$; whence adding 2 units to the characteristic of this last to obtain the logarithm of the proposed, we have $\log 675437 = 5.82959$.

209. We proceed next to the second of the proposed questions, viz. *A logarithm being given, to find the number which corresponds to it.*

1. To find the number corresponding to the logarithm 2.10449. The decimal part of this logarithm is contained in the tables; in the left hand column and on the same line with it according to the arrangement of the tables, in which there are but three places

of figures in the column of numbers, we find 127, and at the top of the table directly over it we find 2; the characteristic of the logarithm being 2, we have therefore 127.2 for the number corresponding to the proposed.

2. To find the number corresponding to the logarithm 3.42674. This logarithm is not found in the tables; it is comprised however between 3.42667 the logarithm of 2671, and 3.42684 that of 2672; the difference between these two logarithms is .00017, the difference between the proposed and 3.42667 is .00007; we have then the following proportion:

$$.00017 : 1 : : .00007 : .41 \text{ nearly.}$$

The number corresponding to 3.42674 is, therefore, 2671.41.

3. To find the number corresponding to the logarithm -2.45379. The number corresponding to this logarithm will be comprised, it is evident, between .01 and .001; to obtain this number let us add to -2.45379 a sufficient number of units to make it positive, 5 for example, we have $5-2.45379=2.54621$; the number corresponding to this last is 351.73; but by adding 5 units to the proposed logarithm, we have multiplied the number, to which it belongs, by 100000, whence, dividing 351.73 by 100000, we have .0035173, the number corresponding to the proposed.

4. To find the number corresponding to the logarithm $\bar{3}.86249$. Adding three units to the characteristic, the proposed becomes 0.86249, the number corresponding to which is 7.286; whence, as it is easy to see, the number corresponding to $\bar{3}.86249$ is .007286

SECTION XXIV.—Application of the Theory of Logarithms.

MULTIPLICATION AND DIVISION.

1. Let it be required to multiply 872 by .097.

$$\begin{aligned} \log 872 &= 2.94052 \\ \log .097 &= \bar{2}.98677 \\ \hline \log 84.584 \ \textit{Ans.} \qquad & \quad 1.92729 \end{aligned}$$

2. Let it be required to multiply .857 by .0093.

$$\log .857 = \bar{1}.93298$$
$$\log .0093 = \bar{3}.96848$$

log .00797 *Ans.* $\bar{3}.90146$

3. Let it be required to divide 5672 by .0037.

$$\log 5672 = 3.75374$$
$$\log .0037 = \bar{3}.56820$$

log 1533000 *Ans.* 6.18554

4. Let it be required to divide .053 by 797.

$$\log .053 = \bar{2}.72428 = \bar{3} + 1.72428$$
$$\log 797 = \quad 2.90146$$

log .0000665 *Ans.* $\bar{5}.82282$

To render the subtraction required in this example possible, we change the characteristic $\bar{2}$ into $\bar{3} + 1$, which has the same value; this furnishes a ten to be joined with 7 for the subtraction of 9, the left hand figure of the decimal part. A similar preparation, it is evident, must be made in all cases of the same kind.

FORMATION OF POWERS AND EXTRACTION OF ROOTS.

210. Let it be required to find the 5th power of .125.

$$\log .125 = \bar{1}.09691$$
$$5$$

log .000030519 *Ans.* nearly $\bar{5}.48455$

2. To find the 7th power of .73.

$$\log .73 = \bar{1}.86332$$
$$7$$

log .11047 *Ans.* nearly $\bar{7} + 6.04324 = \bar{1}.04324.$

3. To find the third root of .01356.

The logarithm of .01356 is $\bar{2}.13226$. The negative characteristic $\bar{2}$ of this logarithm is not divisible by 3, the index of the root required, neither can it be joined to the positive part on account of the different sign. If however we add $-1 + 1$ to

the characteristic, which will not alter its value, it becomes $\bar{3}+1$; the negative part is then divisible by 3, and the 1 being positive may be joined to the fractional part, we have then

$$\log .01356 = \bar{2}.13226 = \bar{3} + 1.13226;$$

whence dividing by 3, we have

$$\bar{1}.37742 = \log .23846 \quad \textit{Ans. } \text{nearly.}$$

In all cases, if the negative characteristic is not divisible by the index of the root required, it must be made so in a similar manner.

ARITHMETICAL COMPLEMENT.

211. The arithmetical complement of a logarithm is the difference between this logarithm and 10; thus the arithmetical complement of 3.472584 is $10 - 3.472584 = 6.527416$. The arithmetical complement of a logarithm is obtained *by subtracting the right hand figure, if it be significant, from* 10, *and the others from* 9.

Let it be proposed to find the value of x in the expression

$$x = l - l' + l'' - l''' - l''''$$

l, l', l'' . . being logarithms; this expression, it is evident, may be put under the form

$$x = l + (10 - l') + l'' + (10 - l''') + (10 - l'''') - 30;$$

that is, to find the value of x, *we take the sum of the logarithms to be added and the complements of the logarithms to be subtracted, from this sum subtract as many times* 10, *as there are complements employed.*

Thus when there are several multiplications and divisions to be performed together, by using the complements of the logarithms of the divisors the whole may be reduced to the addition of logarithms.

EXAMPLES.

1. To find the value of x in the expression

$$x = \left(\frac{3.75 \times 73 \times .056}{1.7498 \times 125.13}\right)^{\frac{4}{3}}$$

log 3.75	0.57403
log 73	1.86332
log .056	$\bar{2}.74819$
log 1.7498 Comp.	9.75701
log 125.13 Comp.	7.90264
	18.84519

Subtracting next 20 from the characteristic, and taking $\frac{1}{3}$ of the remainder, we have $\bar{2}.07532 = \log .011803$ Ans.

2. To find the value of x in the expression

$$x = \left(\frac{132 \times (7.356)^9}{(3.25)^{\frac{5}{2}}}\right)^{\frac{1}{4}} \qquad \text{Ans. } 144.5972.$$

PROPORTIONS.

212. Let it be required to find the fourth term of the proportion, of which the numbers 963, 1279, 8.7, are the first three terms.

log 1279	3.10687
log 8.7	0.93952
log 963 Comp.	7.01637
log 11.555 *Ans.* nearly	1.06276

From the proportion $a : b :: c : d$, we have $\frac{a}{b} = \frac{c}{d}$;

whence $\log a - \log b = \log c - \log d$,

therefore $\log a \,.\, \log b : \log c \,.\, \log d$,

that is, *if four numbers form a proportion, their logarithms will form an equidifference.*

EXPONENTIAL EQUATIONS.

213. We have already explained a method for finding the value of x in the equation $a^x = b$, from which the theory of logarithms is derived; but a table of logarithms being once constructed, there is nothing to prevent its use in the solution of equations of this kind.

Let it be required to find the value of x in the equation $3^x = 15$.

Taking the logarithms of both sides, we have

$$x \log 3 = \log 15;$$

whence $$x = \frac{\log 15}{\log 3} = \frac{1.17609}{.47712} = 2.464 +$$

The division required in this example may be performed, it is easy to see, by subtracting the logarithm of .47712 from that of 1.17609, as in the case of any other numbers.

PROGRESSION BY QUOTIENT.

214. Logarithms are particularly useful in the solution of questions in progression by quotient.

Let it be proposed to find the 20th term in the progression

$$1 : \frac{3}{2} : \frac{9}{4} : \frac{27}{8} \ldots$$

Putting u for the last term of a progression by quotient, we have, art. 176,

$$u = aq^{n-1}; \text{ whence, } \log u = \log a + (n-1) \log q.$$

We have, therefore, for the 20th term in the progression proposed

$$\log u = \log 1 + 19 (\log 3 - \log 2) = 19 (\log 3 - \log 2)$$

the term required will therefore be 2216.84 to within .01.

Let it be required next to insert between the numbers 2 and 15 fifty mean proportionals; we have for the ratio, art. 184

$$q = \sqrt[m+1]{\frac{b}{a}}; \text{ whence } \log q = \frac{\log b - \log a}{m+1};$$

in the question proposed, we have, therefore,

$$\log q = \frac{\log 15 - \log 2}{51},$$

or, performing the calculations, we obtain

$$q = 1.040286.$$

215. Let it be required to find the sum of the first ten terms in the progression $\div\div$ 5 . 15 . 45 . . . ; we have, art. 178,

$$S = \frac{a(q^n - 1)}{q - 1}; \text{ whence}$$

$$\log S = \log a + \log (q^n - 1) - \log (q - 1).$$

Applying this formula to the proposed question, we have

$$\log S = \log 5 + \log (3^{10} - 1) - \log (3 - 1).$$

Calculating 3^{10} by logarithms, we have

$$\log 3^{10} = 10 \times \log 3,$$

from which we obtain $3^{10}=59048$;
whence $\log S=\log 5+\log(59048-1)-\log 2$,
or, performing the calculations, we obtain 147620 for the sum required.

Let it be proposed next to find the number of terms in the progression of which the first term is 3, the ratio 2, and the last term 6144.

From the formula $u=aq^{n-1}$ we have

$$\log u=\log a+(n-1)\log q;$$

whence $$n=1+\frac{\log u-\log a}{\log q}.$$

Applying this formula to the proposed question, we have

$$n=1+\frac{\log 6144-\log 3}{\log 2}=1+11=12.$$

216. Let us take next the progression

$$\because\!\because a:b:c:d:e:f:g\ldots$$

from the nature of the progression, we have

$$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}=\frac{f}{g};$$

whence $$\log\frac{a}{b}=\log\frac{b}{c}=\log\frac{c}{d}=\log\frac{d}{e}\ldots.$$

wherefore, $\log a-\log b=\log b-\log c=\log c-\log d=\ldots$
from this last we have

$$\div \log a\,.\,\log b\,.\,\log c\,.\,\log d\ldots.$$

If, therefore, the numbers a, b, c, d . . form a progression by quotient, their logarithms will form a progression by difference. Logarithms may therefore be defined *a series of numbers in arithmetical progression corresponding term to term to another series of numbers in geometrical progression.* This is the definition of logarithms given in arithmetic.

COMPOUND INTEREST.

217. One of the most important applications of logarithms is to questions upon the interest of money.

Interest is of two kinds, *simple* and *compound.* If interest be paid upon the principal only, it is called *simple interest;* but if

the interest, as it becomes due, be added to the principal, and in terest be paid upon the whole, it is then called *compound* interest.

We have already investigated formulas for simple interest. Let it now be proposed to determine what sum a given principal p will amount to, in a number n of years, at a given rate r at compound interest.

The amount of unity for one year will be $1 + r$; that of p units will be therefore $p(1 + r)$.

For the second year $p(1 + r)$ will be the principal, and its amount will be $p(1 + r)(1 + r)$ or $p(1 + r)^2$.

The original sum p therefore at the end of the second year will amount to $p(1 + r)^2$. In like manner at the end of the third year it will amount to $p(1 + r)^3$; whence putting A for the amount required, we have

$$A = p(1 + r)^n.$$

This is a general formula for compound interest; taking the logarithms of both sides we have,

$$\log A = \log p + n \log(1 + r).$$

Let it be proposed to determine what sum \$30000 will amount to, in 30 years, at 5 per cent. compound interest.

We have $\log A = \log 30000 + 30 \log 1.05$,

whence, we obtain \$129658.27, Ans.

218. The equation $A = p(1 + r)^n$ contains four quantities A, p, r, and n, any one of which may be determined, when the others are known. It gives rise therefore to the four following questions, viz.

1°. *To determine* A, *when* p, r, *and* n *are given, or the principal, rate, and number of years being given, to find the amount.*

This question we have already solved.

2°. *To determine* p *when* A, r, *and* n *are given, or to find what principal put at compound interest will amount to a given sum, in a certain number of years, at a given rate.*

Resolving the general equation with reference to p, we have

$$p = \frac{A}{(1 + r)^n},$$

or by logarithms $\log p = \log A - n \log(1 + r)$.

3°. *To determine* r, *when* A, p, *and* n *are known*, that is, *to find at what rate a given sum must be put at compound interest, in order to amount to another given sum in a given time.*

Resolving the general equation with reference to r, we have

$$(1+r)=\sqrt[n]{\frac{A}{p}},$$

or by logarithms $$\log(1+r)=\frac{\log A-\log p}{n}.$$

Having by means of this last determined the value of $1+r$, that of r will be easily found.

4°. *To determine* n, *when* A, p, *and* r *are given*, that is, *to find for what time a given sum must be put at compound interest at a certain rate in order to amount to a given sum.*

Making n the unknown quantity in the general formula, we obtain

$$n=\frac{\log A-\log p}{\log(1+r)}.$$

If it be asked what must be the value of n in order that the sum at interest may be doubled, tripled, &c.; we put in the general formula $A=kp$, k denoting 1, 2, 3 . . . , we thus have

$$kp=p(1+r)^n; \text{ whence } n=\frac{\log k}{\log(1+r)};$$

n is therefore independent of p, that is, whatever the sum put out, it will be doubled, tripled, &c. in the same time.

EXAMPLES.

1. What the amount of \$1000 for 25 years at 5 per cent. compound interest? Ans. \$3386.

2. What will \$600 amount to in 6 years at $4\frac{1}{2}$ per cent. compound interest, supposing the interest to be payable half yearly? Ans. \$783.63.

3. In a certain province there are at present 200000 inhabitants. If the population increases $\frac{1}{50}$ part yearly, what will it be 100 years hence? Ans. 1448927, nearly.

4. How much money must be placed out at compound interest to amount to \$1000 in 20 years, the interest being 5 per cent.? Ans. \$376.89.

5. A sum of \$201.22 is payable 12 years hence without interest. What sum put out at 6 per cent. compound interest will be sufficient to meet the payment at the end of that time?

Ans. \$100.00.

6. The sum of \$500 put out at 5 per cent. compound interest has amounted already to \$900. How long has it been at interest? Ans. 12.04 years.

7. A capital of \$3200 having been at compound interest for 80 years has amounted to \$34050.84, at what rate per cent. was it put out? Ans. 3 per cent.

8. In what time will a principal be doubled at 5 per cent.? In what time will it be tripled at 6 per cent.?

ANNUITIES.

219. An annuity is a sum of money payable yearly for a certain number of years or forever.

Let it be proposed to determine what sum must be put at interest to pay an annuity of b dollars for n years, the interest being reckoned at the rate r compound interest.

According to the rule for compound interest, the amount of the first payment, at the expiration of the n years, will be $b(1+r)^{n-1}$, the amount of the second payment will be $b(1+r)^{n-2}$, that of the third will be $b(1+r)^{n-3}$ . . . . . the last payment will be b. Putting A for the sum placed at interest for the payment of the annuity, its amount at the end of the n years will be $A(1+r)^n$; we shall have therefore
$A(1+r)^n = b(1+r)^{n-1} + b(1+r)^{n-2} + b(1+r)^{n-3} \ldots b$,
but the second member of this equation forms, it is evident, a progression by quotient the ratio of which is $\frac{1}{1+r}$, or, the order of the series being reversed, $1+r$; taking its sum we have

$$A(1+r)^n = \frac{b[(1+r)^n - 1]}{r};$$

whence
$$A = \frac{b[(1+r)^n - 1]}{r(1+r)^n}.$$

This equation gives rise also to four different questions, according as we make A, b, r, or n the unknown quantity. The following examples exhibit particular cases of these questions.

1. A man wishes to purchase an annuity which shall afford him $1500 a year for 12 years. What sum must he deposit in the annuity office to produce this sum, supposing he can be allowed $7\frac{1}{2}$ per cent. interest? Ans. $11602.91.

2. A man purchased an annuity for 15 years for $100000. How much can he draw annually, the interest being reckoned at 5 per cent.? Ans. $9634.22.

3. A man has property to the amount of $34580, which yields him an income of 4 per cent. His annual expenses are $2000. How long will his property last him? Ans. 30 years nearly.

SECTION XXV.—Praxis.

I.—EQUATIONS OF THE FIRST DEGREE.

1. Given $(x+40)^{\frac{1}{2}}=10-x^{\frac{1}{2}}$, to find the value of x.

Squaring both sides of the equation, we have

$$x+40=100-20x^{\frac{1}{2}}+x$$

whence $$x=9.$$

2. Given $(x-16)^{\frac{1}{2}}=8-x^{\frac{1}{2}}$, to find the value of x.

Ans. $x=25$.

3. Given $\dfrac{x^{\frac{1}{2}}+28}{x^{\frac{1}{2}}+4}=\dfrac{x^{\frac{1}{2}}+38}{x^{\frac{1}{2}}+6}$, to find the value of x.

Freeing from denominators and reducing, we have $16=8x^{\frac{1}{2}}$, whence $x=4$.

4. Given $\dfrac{(9x)^{\frac{1}{2}}-4}{x^{\frac{1}{2}}+2}=\dfrac{15+(9x)^{\frac{1}{2}}}{x^{\frac{1}{2}}+40}$, to find the value of x.

Ans. $x=4$.

5. Given $(2+x)^{\frac{1}{2}}+x^{\frac{1}{2}}=\dfrac{4}{(2+x)^{\frac{1}{2}}}$, to find the value of x.

Ans. $x=\frac{2}{3}$.

6. Given $x^{\frac{1}{2}} + (x-9)^{\frac{1}{2}} = \dfrac{36}{(x-9)^{\frac{1}{2}}}$, to find the value of x.

Ans. $x = 25$.

II.—INCOMPLETE EQUATIONS OF THE SECOND DEGREE.

1. Given $\left.\begin{array}{r} x^3 + y^3 = 189 \\ \text{and } x^2y + xy^2 = 180 \end{array}\right\}$ to find the values of x and y.

Adding 3 times the second equation to the first and extracting the third root, we have $x + y = 9$,
but $x^2y + xy^2 = xy(x+y)$, whence $9xy = 180$, and $xy = 20$; subtracting 4 times this last from the equation $x + y = 9$ raised to the square, and extracting the square root of both sides of the remainder, we obtain $x - y = 1$; whence $x = 5$, $y = 4$.

2. Given $\left.\begin{array}{r} x^2 + xy = 12 \\ \text{and } y^2 + xy = 24 \end{array}\right\}$ to find the values of x and y.

Ans. $x = \pm 2$, $y = \pm 4$.

3. Given $\left.\begin{array}{r} x^2 - xy = 54 \\ \text{and } xy - y^2 = 18 \end{array}\right\}$ to find the values of x and y.

Ans. $x = \pm 9$, $y = \pm 3$.

4. Given $\left.\begin{array}{r} x^3 - y^3 = 56 \\ \text{and } x^2y - xy^2 = 16 \end{array}\right\}$ to find the values of x and y.

Ans. $x = 4$ or -2, $y = 2$ or -4.

5. Given $\left.\begin{array}{r} x^2 + y^2 = \dfrac{13}{x-y} \\ \text{and } xy = \dfrac{6}{x-y} \end{array}\right\}$ to find the values of x and y.

Ans. $x = 3$ or -2, $y = 2$ or -3.

6. Given $\left.\begin{array}{r} x^{\frac{2}{3}} + y^{\frac{2}{3}} = 13 \\ \text{and } x^{\frac{1}{3}} + y^{\frac{1}{3}} = \ \ 5 \end{array}\right\}$ to find the values of x and y.

Squaring the second equation

$$x^{\frac{2}{3}} + 2x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} = 25$$

but $$x^{\frac{2}{3}} \qquad\qquad + y^{\frac{2}{3}} = 13$$

whence by subtraction $$2x^{\frac{1}{3}}y^{\frac{1}{3}} = 12$$

subtracting this last from the first equation

$$x^{\frac{2}{3}} - 2x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} = 1$$

whence $$x^{\frac{1}{3}} - y^{\frac{1}{3}} = \pm 1$$

which compared with the second equation, we obtain

$$x = 27 \text{ or } 8, \; y = 8 \text{ or } 27.$$

7. Given $\left.\begin{array}{l} x + y = 8 \\ \text{and } x^2 + y^2 = 34 \end{array}\right\}$ to find the values of x and y.

Ans. $x = 5$ or 3, $y = 3$ or 5.

8. Given $\left.\begin{array}{l} xy^2 + y = 21 \\ \text{and } x^2y^4 + y^2 = 333 \end{array}\right\}$ to find the values of x and y.

Ans. $x = 2$, $y = 3$.

9. Given $\left.\begin{array}{l} x^{\frac{4}{3}} + y^{\frac{2}{5}} = 20 \\ \text{and } x^{\frac{2}{3}} + y^{\frac{1}{5}} = 6 \end{array}\right\}$ to find the values of x and y.

Ans. $x = \pm 8$ or $\pm\sqrt{8}$, $y = 32$ or 1024.

10. Given $\left.\begin{array}{l} x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 19 \\ \text{and } x^2 + xy + y^2 = 133 \end{array}\right\}$ to find the values of x and y.

Dividing the second equation by the first, we have

$$x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 7;$$

adding this last to the first and dividing by 2, we obtain $x + y = 13$; subtracting it from the first, dividing by 2 and squaring both sides of the result, we have $xy = 36$; comparing the equations thus obtained, we have $x = 9$ or 4, $y = 4$ or 9.

11. Given $\left.\begin{array}{l} \dfrac{xy}{x^{\frac{1}{2}}y^{-\frac{1}{2}}} = 48 \\ \text{and } \dfrac{xy}{x^{\frac{1}{2}}} = 24 \end{array}\right\}$ to find the values of x and y.

Ans. $x = 36$, $y = 4$.

12. Given $\left.\begin{array}{l} x^4 - y^4 = 369 \\ \text{and } x^2 - y^2 = 9 \end{array}\right\}$ to find the values of x and y.

Ans. $x = \pm 5$, $y = \pm 4$.

13. Given $\left.\begin{array}{l} x^2y + xy^2 = 6 \\ \text{and } x^3y^2 + x^2y^3 = 12 \end{array}\right\}$ to find the values of x and y.

Ans. $x = 2$ or 1, $y = 1$ or 2.

III.—COMPLETE EQUATIONS OF THE SECOND DEGREE.

1. Given $x^{\frac{6}{5}} + x^{\frac{3}{5}} = 756$, to find the values of x.

Completing the square, $x^{\frac{6}{5}} + x^{\frac{3}{5}} + \frac{1}{4} = \frac{3025}{4}$.

extracting the root $\qquad x^{\frac{3}{5}} + \frac{1}{2} = \pm \frac{55}{2}$;

from which we obtain $x = 243$ or $(-28)^{\frac{5}{3}}$.

2. Given $x^3 - x^{\frac{3}{2}} = 56$, to find the values of x.

Ans. $x = 4$, or $(-7)^{\frac{2}{3}}$.

3. Given $3x^{\frac{5}{3}} + x^{\frac{5}{6}} = 3104$, to find the values of x.

Ans. $x = 64$, or $(-\frac{97}{3})^{\frac{6}{5}}$.

4. Given $x^{\frac{5}{2}} + x^{\frac{3}{2}} = 6x^{\frac{1}{2}}$, to find the values of x.

Ans. $x = 2$, or -3.

5. Given $x^{\frac{3}{2}} - x = 2x^{\frac{1}{2}}$, to find the values of x.

Ans. $x = 4$ or 1.

6. Given $x + 5 - (x+5)^{\frac{1}{2}} = 6$, to find the values of x.

Completing the square, $x + 5 - (x+5)^{\frac{1}{2}} + \frac{1}{4} = \frac{25}{4}$,

extracting the root $\qquad (x+5)^{\frac{1}{2}} - \frac{1}{2} = \pm\frac{5}{2}$;

from which we obtain $\qquad x = 4$, or -1.

7. Given $(x+12)^{\frac{1}{2}} + (x+12)^{\frac{1}{4}} = 6$, to find the values of x.

Ans. $x = 4$, or 69.

8. Given $x + 16 - 7(x+16)^{\frac{1}{2}} = 10 - 4(x+16)^{\frac{1}{2}}$, to find the values of x.

Ans. $x = 9$, or -12.

9. Given $x^2 + (5x + x^2)^{\frac{1}{2}} = 42 - 5x$, to find the values of x.

Ans. $x = 4$, or -9.

10. Given $x^2 - 2x + 6(x^2 - 2x + 5)^{\frac{1}{2}} = 11$, to find the values of x.

Adding 5 to each member

$$x^2 - 2x + 5 + 6(x^2 - 2x + 5)^{\frac{1}{2}} = 16,$$

completing the square

$$x^2 - 2x + 5 + 6(x^2 - 2x + 5)^{\frac{1}{2}} + 9 = 25,$$

extracting the root and reducing, we obtain

$$x = 1, \text{ or } \pm 2\sqrt{15}.$$

11. Given $9x - 4x^2 + (4x^2 - 9x + 11)^{\frac{1}{2}} = 5$, to find the values of x.

Ans. $x = 2$, or $\frac{1}{4}$.

12. Given $(x^2 + 5)^2 - 4x^2 = 160$, to find the values of x.

Ans. $x = 3$, or $\sqrt{-15}$.

13. Given $(x^2-7x)+(x^2-7x+18)^{\frac{1}{2}}=24$, to find the values of x. Ans. $x=9$, or -2.

14. Given $2x^2+3x-5(2x^2+3x+9)^{\frac{1}{2}}+3=0$, to find the values of x. Ans. $x=3$, or $-4\frac{1}{2}$.

15. Given $x+(x+6)^{\frac{1}{2}}=2+3(x+6)^{\frac{1}{2}}$, to find the values of x. Ans. $x=10$, or -2.

16. Given $\dfrac{x+x^{\frac{1}{2}}}{x-x^{\frac{1}{2}}}=\dfrac{x^2-x}{4}$, to find the values of x.

Ans. $x=4$, or 1.

17. Given $\left(x+\dfrac{8}{x}\right)^2+x=42-\dfrac{8}{x}$, to find the values of x.

Ans. $x=4$, or 2.

18. Given $\dfrac{x^2}{(x^2-4)^2}+\dfrac{6}{x^2-4}=\dfrac{351}{25x^2}$, to find the values of x.

Ans. $x=\pm 3$.

19. Given $[(x-2)^2-x]^2-(x-2)^2=90-x$, to find the values of x. Ans. $x=6$, or -1.

20. Given $\left.\begin{array}{l} 4xy=96-x^2y^2 \\ \text{and } x+y=6 \end{array}\right\}$ to find the values of x and y.

From the first equation $x^2y^2+4xy=96$, completing the square and extracting the root $xy=8$, or -12.

Ans. $x=4$ or 6, $y=2$ or 4.

21. Given $\left.\begin{array}{r} x^2y^4-7xy^2-945=765 \\ \text{and } xy-y \quad =12 \end{array}\right\}$ to find the values of x and y.

Ans. $x=5$, $y=3$.

22. Given $\left.\begin{array}{r} x^2+x+y=18-y^2 \\ \text{and } xy=6 \end{array}\right\}$ to find the values of x and y.

From the first equation	$x^2+y^2+x+y=18$
from the second	$2xy=12$
by addition	$x^2+2xy+y^2+x+y=30$
or	$(x+y)^2+(x+y)=30$
whence	$x=3$ or 2, $y=2$ or 3.

23. Given $\left.\begin{array}{r} x^2+y^2-x-y=78 \\ \text{and } xy+x+y=39 \end{array}\right\}$ to find the values of x and y.

Ans. $x=9$ or 3, $y=3$ or 9.

24. Given $\left.\begin{array}{l} x^2+3x+y=73-2xy \\ \text{and } y^2+3y+x=44 \end{array}\right\}$ to find the values of x and y.

Ans. $x=4$ or 16, $y=5$ or -7.

25. Given $\left.\begin{array}{l} x - 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y = x^{\frac{1}{2}} - y^{\frac{1}{2}} \\ \text{and } x^{\frac{1}{2}} + y^{\frac{1}{2}} = 5 \end{array}\right\}$ to find the values of x and y.

From the first equation we have $(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2 - (x^{\frac{1}{2}} - y^{\frac{1}{2}}) = 0$.

Ans. $x = 9$, $y = 4$.

26. Given $\left.\begin{array}{l} x^2 + 2xy + y^2 + 2x = 120 - 2y \\ \text{and } xy - y^2 = 8 \end{array}\right\}$ to find the values of x and y.

Ans. $y = 4$ or 1, $x = 6$ or 9.

27. Given $\left.\begin{array}{l} x + 4x^{\frac{1}{2}} + 4y = 21 + 8y^{\frac{1}{2}} + 4x^{\frac{1}{2}}y^{\frac{1}{2}} \\ \text{and } x^{\frac{1}{2}} + y^{\frac{1}{2}} = 6 \end{array}\right\}$ to find x and y.

Ans. $x = 25$, $y = 1$.

IV.—MISCELLANEOUS QUESTIONS.

1. A farmer has a stack of hay, from which he sells a quantity, which is to the quantity remaining in the proportion of 4 to 5. He then uses 15 loads and finds that he has a quantity left, which is to the quantity sold as 1 to 2. How many loads did the stack at first contain? Ans. 45.

2. A person engaged to reap a field of 35 acres, consisting partly of wheat and partly of rye. For every acre of rye he received 5 shillings; and what he received for an acre of wheat, augmented by one shilling, is to what he received for an acre of rye as 7 to 3. For his whole labor he received £13. Required the number of acres of each sort.

Ans. 15 acres of wheat and 20 of rye.

3. A person put out a certain sum at interest for $6\frac{1}{2}$ years at 5 per cent. simple interest, and found that if he had put out the same sum for 12 years and 9 months at 4 per cent. he would have received \$185 more. What was the sum put out?

Ans. \$1000.

4. Two persons, A and B, were partners. A's money remained in the firm 6 years, and his gain was one-fourth of his principal, and B's money, which was £50 less than A's, had been in the firm 9 years, when they dissolved partnership, and it appeared that if B had gained £6. 5s. less, his gain and princi-

pal would have been to A's gain and principal as 4 to 5. What was the principal of each? Ans. £200 and £150.

5. The crew of a ship consisted of her complement of sailors and a number of soldiers. Now there were 22 seamen to every three guns and ten over. Also the whole number of hands was 5 times the number of soldiers and guns together. But after an engagement, in which the slain were one-fourth of the survivors, there wanted 5 to be 13 men to every 2 guns. Required the number of guns, soldiers, and sailors.

Ans. 90 guns, 55 soldiers, and 670 sailors.

6. A shepherd in time of war was plundered by a party of soldiers, who took $\frac{1}{4}$ of his flock and $\frac{1}{4}$ of a sheep; another party took from him $\frac{1}{3}$ of what he had left and $\frac{1}{3}$ of a sheep; then a third party took $\frac{1}{2}$ of what now remained and $\frac{1}{2}$ of a sheep. After which he had but 25 sheep left. How many had he at first? Ans. 103.

7. A trader maintained himself for 3 years at the expense of $50 a year; and in each of those years augmented that part of his stock, which was not so expended by one-third thereof. At the end of the third year his original stock was doubled. What was his stock? Ans. $740.

8. When wheat was 5 shillings a bushel and rye 3 shillings, a man wanted to fill his sack with a mixture of rye and wheat for the money he had in his purse. If he bought 7 bushels of rye, and laid out the rest of his money in wheat, he would want two bushels to fill his sack; but if he bought 6 bushels of wheat, and filled his sack with rye, he would have 6 shillings left. How must he lay out his money and fill his sack?

Ans. He must buy 9 bushels of wheat, and 12 bushels of rye.

9. In one of the corners of a garden there is a rectangular fish-pond, whose area is one-ninth part of the area of the garden; the garden is rectangular and its periphery exceeds that of the fish-pond by 200 yards. Also if the greater side be increased by 3 yards and the other by 5 yards, the garden will be enlarged

by 645 square yards. Required the periphery of the garden, and the length of each side.

Ans. The periphery is 300 yards, and the sides are 90 and 60 yards respectively.

10. A sets out express from C towards D, and three hours afterwards B sets out from D towards C, travelling 2 miles an hour more than A. When they meet it appears that the distances they have travelled are in the proportion of 13 to 15; but had A travelled 5 hours less and B gone 2 miles an hour more, they would have been in the proportion of 2 to 5. How many miles did each go per hour, and how many hours did they travel before they met?

Ans. A went 4, and B 6 miles an hour, and they travelled 10 hours after B set out.

11. There is a number consisting of two digits, which being multiplied by the digit on the left hand, the product is 46; but if the sum of the digits be multiplied by the same digit, the product is only 10. Required the number. Ans. 23.

12. A detachment of soldiers from a regiment being ordered to march on a particular service, each company furnished four times as many men as there were companies in the regiment; but these being found to be insufficient, each company furnished 3 more men; when their number was found to be increased in the ratio of 17 to 16. How many companies were there in the regiment? Ans. 12.

13. A farmer has two cubical stacks of hay. The side of one is three yards longer than the side of the other; and the difference of their contents is 117 solid yards. Required the side of each. Ans. 5 and 2 yards respectively.

14. A and B purchased a farm containing 900 acres of land, at the rate of \$2 an acre, which they paid equally between them; but on dividing the same, A got that part of the farm, which contained the best of the improvements, and agreed to pay 45 cents an acre more than B. How many acres had each, and at what price? Ans. A had 400 acres at \$2,25 an acre, and B 500 acres at \$1,80 an acre.

SECTION XXVI. — General Theory of Equations.

221. The equations thus far considered are of the first and second degrees only. Those of the third degree come next in order. We now proceed, however, to develop the general theory of equations.

No general formulas can be given for the solution of equations of a degree higher than the fourth. And the attention of mathematicians has been directed chiefly to the solution of numerical equations, that is, to those which arise from the algebraic translation of a problem in which the given things are particular numbers. We shall give an elementary view of the principles by means of which this object has been successfully accomplished.

222. In the numerical operations required in the solution of equations of this kind, particularly that of division, certain simplifications are of great utility. We will first explain them.

DETACHED COEFFICIENTS.

1. To multiply $x^3 - 3x^2 + 3x - 1$ by $x^2 - 2x + 1$.

The operation may be abridged by first performing the multiplication upon the coefficients detached from the letters, and afterwards annexing the letters raised to the proper powers.

Commencing with the coefficients the work will stand thus:

$$\begin{array}{rrrrrr}
1 & -3 & +3 & -1 & & \\
1 & -2 & +1 & & & \\
\hline
1 & -3 & +3 & -1 & & \\
 & -2 & +6 & -6 & +2 & \\
 & & 1 & -3 & +3 & -1 \\
\hline
1 & -5 & +10 & -10 & +5 & -1.
\end{array}$$

The product of x^3 by x^2 is x^5; the highest power of x in the product will be, therefore, x^5; and since from the arrangement, the powers of this letter go on decreasing by unity, we shall have, it is evident, for the powers,

$$x^5, x^4, x^3, x^2, x.$$

Annexing these to the coefficients, the required product will be

$$x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1.$$

2. If any powers are wanting in either of the factors, they must be supplied by writing them with 0 as a coefficient. Thus, let it be required to multiply

$$3x^4 - 7x^3y + 8x^2y^2 - 5y^4 \text{ by } 2x^2 - 3xy + y^2$$

Here a term xy^3 in the multiplicand is wanting, which must be supplied, thus, $0xy^3$. The operations upon the coefficients will then be as follows:

$$\begin{array}{rrrrrrr}
3- & 7+ & 8+ & 0- & 5 & & \\
2- & 3+ & 1 & & & & \\
\hline
6- & 14+ & 16+ & 0- & 10 & & \\
- & 9+ & 21- & 24- & 0+ & 15 & \\
 & & 3- & 7+ & 8+ & 0- & 5 \\
\hline
6- & 23+ & 40- & 31- & 2+ & 15- & 5,
\end{array}$$

The powers of x go on decreasing by unity, and those of y increasing by unity. Supplying these, the product required will be

$$6x^6 - 23x^5y + 40x^4y^2 - 31x^3y^3 - 2x^2y^4 + 15xy^5 - 5y^6$$

3. Multiply $5x^3 - 3ax^2 + 5a^2x - a^3$ by $a^2 + 3ax + 5x^2$. In this case we reverse the order of the terms in the multiplier, so that the arrangement may be the same as in the multiplicand. The operation performed upon the coefficients, as above, will give

$$25 - 0 + 21 + 7 + 2 - 1.$$

And the required product will be

$$25x^5 + 21x^3a^2 + 7x^2a^3 + 2xa^4 - a^5.$$

4. Multiply $2a^3 - 3ab^2 + 5b^3$ by $2a^2 - 5b^2$.

Ans. $4a^5 - 16a^3b^2 + 10a^2b^3 + 15ab^4 - 25b^5$.

5. Multiply $x^4 - ax^3 + a^2x^2 - a^3x + a^4$ by $x + a$.

Ans. $x^5 + a^5$.

The process above is called *Multiplication by Detached Coefficients*. The examples, art. 24, will serve as an additional exercise.

223. The process of division may, in like manner, be

abridged by first performing the operation upon the coefficients detached from the letters, and then supplying the letters.

1. Let it be required, for example, to divide $a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 5ax^4 - x^5$ by $a^2 - 2ax + x^2$.

The operation upon the coefficients will be as follows:

$$\begin{array}{l|l} 1-5+10-10+5-1 & 1-2+1 \\ \underline{1-2+\ 1} & \overline{1-3+3-1} \\ \quad -3+\ 9-10 & \\ \quad \underline{-3+\ 6-\ 3} & \\ \qquad\quad 3-\ 7+5 & \\ \qquad\quad \underline{3-\ 6+3} & \\ \qquad\qquad -\ 1+2-1 & \\ \qquad\qquad \underline{-\ 1+2-1.} & \end{array}$$

The coefficient of the quotient will be $1 - 3 + 3 - 1$. And, in order to supply the letters, we take the quotient of the letters in the first term of the dividend by those of the divisor; thus, a^5 divided by a^2 gives a^3. The letters in the first term of the quotient will then be a^3, and in the succeeding terms they will follow, it is evident, the law of the dividend. The quotient required will then be $a^3 - 3a^2x + 3ax^2 - x^3$.

2. Divide $6a^4b^2 + 3a^3b^3 - 4a^2b^4 + b^6$ by $3a^3b - 2ab^3 + b^4$.

The operations upon the coefficients will stand thus:

$$\begin{array}{l|l} 6+3-4+0+1 & 3+0-2+1 \\ \underline{6+0-4+2} & \overline{2+1} \\ \qquad 3+0-2+1 & \\ \qquad \underline{3+0-2+1.} & \end{array}$$

Supplying the letters we shall have for the quotient, $2ab + b^2$.

Before commencing the operations, the dividend and divisor should, it is evident, be arranged both in reference to the same letter. The process is called *Division by Detached Coefficients.*

SYNTHETIC DIVISION.

224. The operation for finding the coefficients of the quotient may be still further abridged.

In the ordinary process of division, we multiply the divisor by each term of the quotient as it is found, and subtract successively the partial products from the dividend. The effect, it is evident, will be the same, if we change the signs of the divisor and add the partial products to the dividend. Thus, in the first example above, if we change the signs of the divisor, and then find the terms of the quotient by the first term of the divisor with its sign unchanged, the partial products may be added, and the work will stand as follows:

$$\begin{array}{l|l}
1-5+10-10+5-1 & -1+2-1 \\ \cline{2-2}
\underline{-1+2-1\qquad\quad} & 1-3+3-1 \\
\quad -3+9-10 & \\
\quad \underline{\ \ 3-6+3\qquad} & \\
\qquad\quad 3-7+5 & \\
\qquad \underline{-3+6-3\qquad} & \\
\qquad\qquad -1+2-1 & \\
\qquad\qquad \underline{\ \ \ 1-2+1.} &
\end{array}$$

In this operation it is easy to see that the terms $+9-10$ in the second partial dividend, $-7+5$ in the third, and $+2-1$ in the fourth, may be omitted; and the first term in each partial dividend found by adding all the terms in each column as the work proceeds. With this modification the work will stand thus:

$$\begin{array}{l|l}
1-5+10-10+5-1 & -1+2-1 \\ \cline{2-2}
\underline{-1+2}-1 & 1-3+3-1 \\
\quad -3 & \\
\quad \underline{+3+6}+3 & \\
\qquad\quad 3 & \\
\qquad \underline{-3+6}-3 & \\
\qquad\qquad -1 & \\
\qquad\qquad \underline{+1-2+1} & \\
\qquad\qquad\ \ 0\quad 0\quad 0. &
\end{array}$$

In this process there is liability to error in the signs of the quotient, in consequence of the necessity of finding each term

of the quotient by means of the first term of the divisor with its sign unchanged. To avoid this liability, recollecting that the first term in each successive dividend is always cancelled by the product of the first term of the divisor by the corresponding term of the quotient, we retain the first term of the divisor with its sign unchanged, and change all the rest. The operation will then stand thus:

$$\begin{array}{rrrrrr|l}
1 & -5 & +10 & -10 & +5 & -1 & 1+2-1 \\
 & 2 & -1 & & & & \overline{1-3+3-1} \\
\cline{2-2}
 & -3 & & & & & \\
 & & -6 & +3 & & & \\
\cline{3-3}
 & & -3 & & & & \\
 & & & +6 & -3 & & \\
\cline{4-4}
 & & & -1 & & & \\
 & & & & -2 & +1 & \\
\cline{5-6}
 & & & & 0 & 0 &
\end{array}$$

The work may be written more concisely thus:

$$\begin{array}{lr|rrrrrr}
 & 1 & 1 & -5 & +10 & -10 & +5 & -1 \\
 & 2 & & 2 & -6 & +6 & -2 & \\
 & -1 & & & -1 & +3 & -3 & +1 \\
\cline{3-8}
\text{First term of Dividends,} & & & -3 & +3 & -1 & 0 & 0 \\
\cline{3-8}
\text{Quotient,} & & 1 & -3 & +3 & -1 & &
\end{array}$$

The divisor is placed at the left of the dividend in a vertical column. Beneath, in a horizontal line, are placed the first terms of the successive partial dividends; and under the whole is written the quotient also in a horizontal line. The partial products are written under the terms of the dividend to which they belong, in a diagonal line from the left downwards toward the right.

2. Divide $2a^7 - 6a^4 + 4a^3 - 7a^2 + 9$
by $2a^3 + 6a^2 - 10$.

$$
\begin{array}{r|rrrrrrrr}
2 & 2 & 0 & 0 & -6 & +4 & -7 & 0 & +9 \\
-6 & & -6 & +18 & -54 & +150 & -372 & & \\
0 & & & 0 & 0 & 0 & 0 & 0 & \\
10 & & & & 10 & -30 & +90 & -250 & +620 \\
\hline
 & & -6 & +18 & -50 & +124 & -289 & -250 & +629. \\
\hline
 & 1 & -3 & +9 & -25 & +62. & & &
\end{array}
$$

The operation, it is evident, terminates when the partial products have reached the right hand column. This is the case, in the present example, when the term 62 of the quotient is obtained. And since the columns to the right of this do not, when added, severally reduce to 0, there will be a remainder, of which the sums of these columns respectively will be the coefficients.

Supplying the letters, we shall have, therefore, $a^4 - 3a^3 + 9a^2 - 25a + 62$ for the quotient, with a remainder $-289a^2 - 250a + 629$.

3. Divide $x^6 - 5x^5 + 15x^4 - 24x^3 + 27x^2 - 13x + 5$ by $x^4 - 2x^3 + 4x^2 - 2x + 1$. Ans. $x^2 - 3x + 5$.

4. Divide $x^5 + 2x^4y + 3x^3y^2 - x^2y^3 - 2xy^4 - 3y^5$ by $x^2 + 2xy + 3y^2$. Ans. $x^3 - y^3$.

The process with the modification above is called *Synthetic Division.* The examples, art 39, will furnish an additional exercise for the learner.

General Properties of Equations.

225. Any expression which involves a quantity is called a *function* of that quantity.

Thus, $x^2 + px$, $ax^2 + b$, $(a + x)^3$ are all functions of x.

In like manner, $ax^2 - by^2$, $x^2y + y^2x$, are functions of x and y.

2. A function is usually indicated by some one of the letters, f, F, &c., the quantity or quantities of which the expression is

a function being inclosed in a parenthesis. Thus, $f(x)$ indicates a function of x, $f(x, y)$ a function of x and y.

3. If by $f(x)$ we denote a particular function of x, then $f(a)$ will denote the same function of a. Thus, if the first function is $x^2 + 5x + 6$, the second will be $a^2 + 5a + 6$.

4. It will be recollected that by the *root* of an equation we understand any quantity which, being substituted in the equation, will satisfy its conditions.

5. An equation of the second degree is sometimes called a *quadratic* equation, one of the third degree a *cubic*, and one of the fourth a *bi-quadratic* equation.

6. A complete equation of the nth degree with one unknown quantity, n being an entire and positive number, may be reduced to the form,

$x^n + A x^{n-1} + B x^{n-2} + C x^{n-3} + \quad . \quad . \quad . \quad T x + U = 0$, in which the coefficients A, B, C . . . T, U, are any numbers whatever, positive or negative, entire or fractional.

Every equation of this description, since it is supposed to be derived from a problem with sufficient and properly limited conditions, may be assumed to have at least one root.

We now proceed to investigate the general principles necessary to the solution of numerical equations of any degree.

Divisibility of Equations.

226. 1. Resuming the general equation,

$$x^n + A x^{n-1} + B x^{n-2} + \quad . \quad . \quad . \quad T x + U = 0, \quad (1)$$

if a is a root of the equation, then *the first member is divisible by $x - a$.*

For if the division is not exact, let Q be the quotient, and R the remainder arising from the division by $x - a$; then we have

$$x^n + A x^{n-1} + \quad . \quad . \quad . \quad . \quad T x + U = Q(x - a) + R. \quad (2)$$

But the left hand member of this equation is equal to 0; and

since a is by hypothesis a root of the equation, we have $x = a$ or $x - a = 0$, and the equation (2) reduces to

$$0 = 0 + R, \text{ or } R = 0,$$

that is, there is no remainder, and the division is exact.

2. Conversely, if the first member of the equation (1) is divisible by $x - a$, *then* a *is a root of the equation.* For Q being the quotient arising from the division by $x - a$, the equation returns to

$$Q(x - a) = 0,$$

which is satisfied by the value $x = a$; hence a is a root of the equation.

In the solution of equations we have frequent occasion to ascertain, by trial, whether a particular number is a root of the equation. From the preceding principle it is obvious that this may easily be done by division.

Ex. 1. To determine whether 4 is a root of the equation,

$$x^3 - 9x^2 + 26x - 24 = 0.$$

Dividing by $x - 4$, and performing the operation by synthetic division we have

$$\begin{array}{r|rrrr} 1 & 1 & -9 & +26 & -24 \\ 4 & & 4 & -20 & +24 \\ \hline & 1 & -5 & +6 & \end{array}$$

Ans. 4 is a root, and if the proposed be divided by $x - 4$ the equation which results will be

$$x^2 - 5x + 6 = 0.$$

Ex. 2. To determine whether 5 is a root of the same equation.

Ans. 5 is not a root, since the division by $x - 5$ leaves a remainder of 6.

Ex. 3. Is 2 a root of the equation $x^3 - 7x + 6 = 9$?

Ex. 4. Is 3 a root of the equation $x^3 - 6x^2 + 8x - 16 = 0$?

Number of the Roots.

227. In order to the solution of an equation, we must first determine the number of its roots. An equation of the second degree with one unknown quantity has, we have seen, two roots. We shall now show that every equation with one unknown quantity has as many roots as there are units in the highest power of the unknown quantity, and no more.

Let a be a root of the equation

$$x^n + A x^{n-1} + B x^{n-2} + \quad . \quad . \quad . \quad T x + U = 0;$$

since by the last article this equation is divisible by $x - a$, it returns to

$$(x - a)\ (x^{n-1} + A' x^{n-2} + \quad . \quad . \quad . \quad T' x + U',) = 0,$$

A', &c., being the new coefficients which arise from the division. But this equation is satisfied by $x - a = 0$, or by

$$x^{n-1} + A' x^{n-2} + \quad . \quad . \quad . \quad . \quad T' x + U' = 0.$$

Let b be a root of this last equation, then we have

$$(x - b)\ (x^{n-2} + A'' x^{n-3} + \quad . \quad . \quad . \quad T'' x + U'') = 0,$$

which is satisfied by $x - b = 0$, or by

$$x^{n-2} + A'' x^{n-3} + \quad . \quad . \quad . \quad . \quad T'' x + U' = 0.$$

Continuing the operation, it will be seen that for every new factor obtained, the exponent of x is made one less, and that we shall have finally $x^n + A x^{n-1} + B x^{n-2} + \quad . \quad . \quad . \quad T x + U = (x - a)(x - b)(x - c) \quad . \quad . \quad . \quad . \quad (x - p)$, in which the number of binomial factors, $x - a$, $x - b$, &c., is equal to n or to the number of units in the index of the highest power of the unknown quantity. And since there are as many roots as factors, there will be as many roots as units in the highest power of x, the unknown quantity.

An equation, moreover, cannot have a greater number of roots than there are units in the highest power of x.

Let $V = x^n + A x^{n-1} + B x^{n-2} + \quad . \quad . \quad . \quad T x + U = 0$, the roots of which are $a, b, c \quad . \quad . \quad p$, respectively; then

$$V = (x - a)\ (x - b)\ (x - c) \quad . \quad . \quad . \quad . \quad . \quad (x - p).$$

If it be possible, let a' be another root differing from $a, b, c, \ldots p$; then we shall have

$$V = (a' - a)(a' - b)(a' - c) \ldots (a' - p) = 0;$$

but this equation is impossible, since a' being different from $a, b, c, \ldots p$, no one of the factors of V can be equal to 0.

Every equation, therefore, will have as many roots as there are units in the highest power of the unknown quantity, and no more.

These roots may not, however, be all different. In fact, any number of them may be equal, as a and b, or a, b, and c, &c.

If the equation has two roots, each equal to a, for example, it will be divisible by $(x-a)^2$; if it has three roots, each equal to a, it will then be divisible by $(x-a)^3$, and so on.

A part of the roots, moreover, may be imaginary. But, from what has been said, every equation will have at least one real root.

228. From what has been done, it will be seen that if one or more roots of an equation are known, the reduced equation containing the other roots may easily be found by division.

Ex. 1. One root of the equation $x^3 - 15x^2 + 75x - 125 = 0$, is 5. What is the equation which contains the other roots?

$$\text{By Synthetic Division,}\quad \begin{array}{r|l} 1 & 1 - 15 + 75 - 125 \\ 5 & \quad\;\; 5 - 50 + 125 \\ \hline & 1 - 10 + 25. \end{array}$$

Ans. $x^2 - 10x + 25 = 0$.

Ex. 2. Two roots of the equation $x^4 - 5x^3 - 12x^2 + 76x - 80 = 0$, are 2 and 5. What is the reduced equation which contains the other roots?

OPERATION.

$$\begin{array}{lr|l} \text{1st Division,} & 1 & 1 - 5 - 12 + 76 - 80 \\ & 2 & \quad\; 2 - \;\;6 - 36 + 80 \\ \hline \text{2d Division,} & 1 & 1 - 3 - 18 + 40 \\ & 5 & \quad\; 5 + 10 - 40 \\ \hline & & 1 + 2 - \;\;8 \end{array}$$

Ans. $x^2 + 2x - 8 = 0$.

If, as in the preceding example, the reduced equation is a quadratic, the remaining roots may be found by the methods already explained.

Ex. 3. One root of the equation $x^3 + 3x^2 - 16x + 12 = 0$ is 1; what are the remaining roots? Ans. 2, and -6.

Ex. 4. Two roots of the equation $x^4 - 12x^3 + 48x^2 - 68x + 15 = 0$, are 3 and 5. What are the remaining roots?

Ans. $2 + \sqrt{3}$, and $2 - \sqrt{3}$.

Ex. 5. One root of the equation $x^3 - x^2 - 7x + 15 = 0$, is -3. What are the other two roots?

Ans. $2 + \sqrt{-1}$, and $2 - \sqrt{-1}$.

Coefficients.

229. The roots of an equation are obviously involved in the coefficients. We proceed next to determine the law of the coefficients, or the manner in which they are connected with the roots.

Let it be proposed, then, to form the equation whose roots shall be $a, b, c \;.\;.\;.\;.$ respectively.

The left hand member will be equal, it is evident, to the continued product of $x - a$, $x - b$, $x - c \;.\;.\;.\;.$ Performing the multiplication we have

$$(x-a)(x-b) = x^2 \begin{array}{r|l} -a & x + ab \\ -b & \end{array}$$

$$(x-a)(x-b)(x-c) = x^3 \begin{array}{r|r|l} -a & x^2 + ab & x - abc \\ -b & ac & \\ -c & bc & \end{array}$$

and so on, as in art. 128.

From what has been done we have the following properties, viz.:

1°. The coefficient of the *second* term in the required equation will be *the sum of all the roots* with their signs changed.

2°. The coefficient of the *third* term will be *the sum of the products of every two roots* with their signs changed.

3°. The coefficient of the *fourth* term will be *the sum of the products of every three roots* with their signs changed, and so on.

4°. The *last*, or *absolute term* will be *the product of all the roots* with their signs changed.

That this law is general may be shown, as in art. 129.

From these principles, it follows,

1°. If the coefficient of the second term in any equation is 0, that is, if the second term is wanting, the sum of the positive roots is equal to the sum of the negative roots.

2°. If the signs of the terms of the equation are all positive, the roots are all negative; and if the signs are alternately positive and negative, the roots are all positive.

3°. Every root of an equation is a divisor of the last or absolute term.

1. The following examples exhibit the manner in which the coefficients are derived from the roots.

Ex. 1. Find the equation whose roots are 2, 3, 4 and -5. Indicating the equation it will be

$$(x-2)(x-3)(x-4)(x+5)=0.$$

The coefficients may be found by the principles just demonstrated, or by actual multiplication as follows,

$$\begin{array}{r|l}
-3 & 1-2 \\
 & \quad -3+6 \\ \hline
-4 & 1-5+6 \\
 & \quad -4+20-24 \\ \hline
5 & 1-9+26-24 \\
 & \quad 5-45+130-120. \\ \hline
 & 1-4-19+106-120.
\end{array}$$

Ans. $x^4-4x^3-19x^2+106x-120=0.$

Ex. 2. What is the equation whose roots are 1, 3, and -4?

Ans. $x^3-13x+12=0.$

Ex. 3. What is the equation whose roots are -1, -2, -3, and -5?

Ans. $x^4+11x^3+41x^2+61x+30=0.$

Ex. 4. Find the equation whose roots are 2, 3, 5, and 6.

Ans. $x^4 - 16x^3 + 91x^2 - 216x + 180 = 0$.

2. By means of the first of the preceding principles one of the roots of an equation may be found, when all the rest are determined.

By means of the fourth, the integral roots may all be found. In order to this, we seek among the divisors of the last term those that will satisfy the equation.

Ex. 1. Find the integral roots of the equation $x^3 - 8x^2 + 19x - 12 = 0$. The divisors of the last term are 1, 2, 3, 4, and 6; of these 1, 3, and 4, substituted respectively for x, satisfy the equation, and are, therefore, roots. The equation being of the third degree only, they are all the roots.

Ex. 2. Find the roots of the equation $x^3 - 2x^2 - 5x + 6 = 0$.

Ans. 1, 3, and -2.

Ex. 3. Find the roots of the equation $x^3 - x - 6 = 0$.

Ans. 2 is the only integral root. Depressing the equation by this root, the remaining roots found from the resulting equation are $-1 \pm \sqrt{-2}$.

FORM OF THE ROOTS.

230. The roots of an equation may be entire, fractional, surd or imaginary.

Let there be the equation

$$x^n + Ax^{n-1} + Bx^{n-2} + \ldots\ldots Tx + U = 0,$$

in which the coefficient of the first term is unity, and A, B, &c., entire numbers. To determine whether this equation can have a fractional root:

If it be possible, let the fraction $\frac{a}{b}$, the terms of which are prime to each other, be a root of this equation.

Substituting $\frac{a}{b}$ for x, multiplying both members by b^{n-1}, and transposing we obtain

$$\frac{a^n}{b} = -Aa^{n-1} - Ba^{n-2}b - \ldots\ldots Tab^{n-2} - Ub^{n-1}$$

The right hand member of this equation is an entire quantity, since it is composed of terms each of which is integral. The left hand member must, therefore, be entire, or we shall have a whole number equal to a fraction, which is absurd. Hence *an equation, whose coefficients are all integers and that of the highest power of the unknown quantity equal to unity, cannot have a fractional root.*

It does not follow from this, however, that all the roots are whole numbers. The equation may have other roots, which cannot be expressed in whole numbers or definite fractions, such as surds or imaginary quantities.

231. But *surds and impossible roots enter equations by pairs*, so that if there be one, there will necessarily be two; and if three, there will necessarily be four, and so on.

Let $a + \sqrt{-b}$, for example, be one of the roots of an equation, the coefficients of which are real.

Suppose the equation reduced by division until two only of its roots remain. It will then be a quadratic. And if one of its roots is $a + \sqrt{b}$, the other will necessarily be $a - \sqrt{-b}$. In the same way it may be shown that if there are three surd or imaginary roots, there will necessarily be four, and so on.

From this it follows,

1°. An equation of an even degree may have all its roots imaginary; but if they are not all imaginary, two of them, at least, will be real.

2°. The product of every pair of imaginary roots being of the form, $a^2 + b$, is positive; hence the absolute term of an equation whose roots are all imaginary must be positive.

3°. Every equation of an odd degree has at least one real root; and if there be but one, that root must necessarily have a contrary sign to that of the last term.

4°. Every equation of an even degree whose last term is negative has, at least, two real roots; and if there be but two, one of these will be positive and the other negative.

These principles are illustrated in the following examples.

In forming the equations, the most convenient process will be to multiply together in pairs the factors containing the imaginary roots, and then combine the factors thus obtained.

Ex. 1. Form the equation whose roots are $2+\sqrt{-3}$, $2-\sqrt{-3}$, $3+\sqrt{-1}$, and $3-\sqrt{-1}$.

Ans. $x^4-10x^3+41x^2-82x+70=0$.

Ex. 2. Form the equation whose roots are $3+\sqrt{-5}$, $3-\sqrt{-5}$, and 5. Ans. $x^3-12x^2+50x-84=0$.

Ex. 3. Form the equation whose roots are $5+\sqrt{-1}$, and $5-\sqrt{-1}$. Ans. $x^2-10x+26=0$.

Ex. 4. Form the equation whose roots are 2, $3+\sqrt{-4}$ $3-\sqrt{-4}$, and -5.

Ans. $x^4-3x^3-15x^2+99x-130=0$.

2. An equation which has imaginary roots is divisible by $(x-a+b\sqrt{-1})\ (x-a-b\sqrt{-1})$, or, $(x-a)^2+b^2$; $a+b\sqrt{-1}$, $a-b\sqrt{-1}$, representing any pair of the roots; hence

1°. Every equation may be resolved into rational factors, simple or quadratic.

From what has been done, it is also evident that,

2°. An algebraic equation which has real coefficients is always composed of as many real factors of the first degree as it has real roots, and of as many real factors of the second degree as it has pairs of imaginary roots.

Ex. Form the equation whose roots are $3, -5, 1+\sqrt{-3}$, $1-\sqrt{-3}$. Ans. $x^4-15x^2+38x-60=0$.

1. What are the factors corresponding to the real roots of this equation? Ans. $x-3$, and $x+5$.

2. What is the factor corresponding to the pair of imaginary roots? Ans. x^2-2x+4.

Relation of the Signs to the Roots.

232. In the preceding article we have seen the important relation between the sign of the absolute term of an equation and the form and number of the roots. Let us now examine

the relation of the signs of the terms generally to the roots. Resuming the general equation,

$$x^n + A x^{n-1} + B x^{n-2} + \quad . \quad . \quad . \quad . \quad T x + U = 0, \quad (1)$$

and changing the signs of the alternate terms it becomes

$$x^n - A x^{n-1} + B x^{n-2} - \quad . \quad . \quad . \quad . \pm T x \mp U = 0; \ (2)$$ or

changing all the signs in this last, which will leave the equation identically the same,

$$- x^n + A x^{n-1} - B x^{n-2} + \quad . \quad . \quad . \quad . \mp T x \pm U = 0, (3).$$

Now if a be substituted for x in equation (1), and $-a$ be substituted for x in (2) when n is an even number, or in (3) when n is an odd number, the equations which result will be identically the same. If then a is a root of equation (1) this equation will be verified by this substitution. Hence the equation (2), or (3) as the case may be, will be verified by the substitution of $-a$ for x, and, therefore, $-a$ is a root of the equations (2) and (3).

If, therefore, the signs of the alternate terms in an equation are changed, the signs of all the roots will be changed.

Ex. 1. Form the equations whose roots are $1, 2, 3$; and -1, -2, -3.

Ans. The equations are $x^3 - 6x^2 + 11x - 6 = 0$, and $x^3 + 6x^2 + 11x + 6 = 0$.

Ex. 2. The roots of the equation $x^4 + x^3 - 19x^2 + 11x + 30 = 0$, are $-1, 2, 3$, and -5. What will be the roots if the signs of the alternate terms are changed?

Since the negative roots may be changed into positive by simply changing the signs of the alternate terms, the finding the real roots of an equation is reduced, by the preceding principle, to finding positive roots only.

233. When in an equation the signs continue the same from one term to the next following, there is said to be a *permanence* of signs; and when the signs change from one term to the next following, a *variation* of signs. Thus, in the equation, $x^4 - 3x^3 - 15x^2 + 49x - 12 = 0$, there is one permanence and three variations of signs.

Let $+ + - + - + + +$ be the order of signs in any equation, and let us introduce into this equation a new positive root a. In order to this, we multiply the equation by $x - a$. The operation, so far as the signs are concerned, will be as follows:

$$\begin{array}{ccccccccc}
+ & + & - & + & - & + & + & + & \\
+ & - & & & & & & & \\ \hline
+ & + & - & + & - & + & + & + & \\
 & - & - & + & - & + & - & - & - \\ \hline
+ & \pm & - & + & - & + & \pm & \pm & -
\end{array}$$

in which the ambiguous sign $\pm$ indicates that the sign may be $+$ or $-$, according to the relative magnitude of the partial products with contrary signs united in the terms to which it corresponds.

If now this result be examined with attention, it will be seen that each permanence is changed into an ambiguity by the introduction of the new positive root $+a$. It follows, therefore, that the permanences, take the ambiguous sign as we may, cannot be increased in the final product by the introduction of the new positive root; but, as the number of signs is increased by *one*, the number of variations must be increased by *one*.

In the equation $x - a = 0$, there is one positive root, and one variation. And since, from the reasoning above, the introduction of each new positive root in this equation will produce at least one variation, it follows that the number of positive roots in any equation can never be greater than the number of variations of sign.

By a process altogether similar, it may be shown that the introduction of a new negative root produces at least one new permanence, and that the number of negative roots can never be greater than the number of permanences. Hence, generally, in a complete equation of any degree, the *number of positive roots cannot be greater than the number of variations of sign, nor the number of negative roots greater than the number of permanences.*

Ex. 1. How many permanences and variations of sign are there in the equation whose roots are 2, 3, and -4?

The equation $x-2=0$ gives one variation corresponding to the positive root 2. If we now multiply this by $x-3$, the result, $x^2-5x+6=0$, gives an additional variation corresponding to the new positive root $+3$. Multiplying again by $x+4$, the result, $x^3-x^2-14x+24=0$, gives, as before, two variations corresponding to the positive roots 2, and 3, and one permanence corresponding to the negative root -4.

Ex. 2. How many permanences and variations in the equation whose roots are 2, -3, and -5?

Ans. The equation is $x^3+6x^2-x-30=0$, exhibiting one variation and two permanences.

The whole number of permanences and variations taken together, it is evident, will be equal to the degree of the equation. If all the roots, therefore, are real, the number of positive roots will be equal to the number of variations, and the number of negative roots will be equal to the number of permanences.

234. In what precedes, the equation is supposed to be complete. If there are missing terms their place must be supplied with the coefficient 0. Any sign may be given to this coefficient without affecting the roots of the equation.

Let there be the equation $x^4-5x^2+8x-6=0$. In its present form this equation exhibits variations only. It would appear, therefore, that it can have no negative roots. But if we supply the missing term it becomes

$$x^4\pm 0x^3-5x^2+8x-6=0.$$

Now which ever sign we take with the 0 coefficient, the equation will present one permanence. It may have, therefore, one negative root.

The preceding principle enables us to determine the number of positive and negative roots a proposed equation may have, an object of importance in the research for the roots.

Ex. 1. The equation $x^5-3x^4-5x^3+15x^2+4x-12=0$, has five real roots. How many of them are positive?

Ex. 2. The equation $x^4 - 3x^3 - 15x^2 + 49x - 12 = 0$, has four real roots. How many of them are negative?

Ex. 3. The equation $x^4 - 3x^3 - 15x^2 + 99x - 130 = 0$, has (art. 231) two real roots. What are their signs?

235. By means of the above rule we are sometimes also enabled, when there are missing terms, to detect the existence of imaginary roots in an equation.

Thus, let there be the equation $x^2 + 30 = 0$.

Supplying the missing term, it becomes

$$x^2 \pm 0x + 30 = 0.$$

If the upper sign be taken with the coefficient 0, there will be no variations; hence there can be no positive roots. And if the lower sign be taken, there will be no permanences; hence there can be no negative roots. The two roots of the equation will therefore be imaginary.

Let there be next the equation $x^3 + 10x + 5 = 0$.

Supplying the missing term, it becomes

$$x^3 \pm 0x^2 + 10x + 5 = 0.$$

Here, if the upper sign of the 0 coefficient be taken, the equation exhibits only permanences; it can have, therefore, no positive root. If the lower sign be taken there will be but one permanence, and, therefore, there can be but one negative root. Thus it appears that the equation must have two imaginary roots.

Transformation of Equations.

236. When the solution of an equation is difficult, the work may sometimes be accomplished with more facility by aid of another equation, the roots of which shall bear to those of the proposed some given relation. The roots of the latter being found, those of the former will then be readily deduced from them.

It may, therefore, be required to change a proposed equation into another, the roots of which shall bear to those of the former a given relation. One of the most simple, as well as useful, of

$= 0$?

these transformations is to change the equation into another, the roots of which shall be greater or less than those of the proposed by a given quantity.

To effect the transformation required, it will be sufficient, it is evident, to substitute for x in the proposed x diminished or increased by the desired quantity. The resulting equation will be of the same form with the proposed, the roots of which will be of the required dimensions.

Let it be required, for example, to find an equation whose roots shall be less by 2 than those of the equation

$$x^2 - 8x + 7 = 0. \quad (1)$$

Substituting $x + 2$ for x, the equation becomes

$$(x + 2)^2 - 8(x + 2) + 7 = 0;$$

or developing and reducing,

$$x^2 - 4x - 5 = 0. \ (2)$$

Resolving equation (1) its roots are 7 and 1. Resolving equation (2) its roots are 5 and -1, or less by 2 than those of the proposed, as was required.

Ex. 2. Let it be required to find next an equation whose roots shall be greater by 2 than those of the equation

$$x^3 - 6x^2 + 11x - 6 = 0.$$

Ans. $x^3 - 12x^2 + 47x - 60 = 0.$

237. The process above is laborious, especially in the higher equations. Let us see if one more simple can be found.

Resuming the general equation

$$x^n + Ax^{n-1} + Bx^{n-2} + \quad . \quad . \quad . \quad . \quad Tx + U = 0,$$

let us make $x = u + x'$, u being a new unknown, and x' an indeterminate, to which any value, positive or negative, may be assigned at pleasure.

Substituting $u + x'$, or, which is the same thing, $x' + u$ for x in the proposed, it becomes

$$(x' + u)^n + A(x' + u)^{n-1} + B(x' + u)^{n-2} \ . \ . \ . \quad T(x' + u)$$
$$+ U = 0, \ (1)$$

or developing the several terms by the binominal formula, and arranging with reference to the ascending powers of u,

$$\left.\begin{array}{l} x'^{n} \\ A\,x'^{n-1} \\ B\,x'^{n-2} \\ \cdot\;\cdot\;\cdot \\ T\,x' \\ U \end{array}\right| u^{\circ} + \left.\begin{array}{r} n\,x'^{n-1} \\ (n-1)\,A\,x'^{n-2} \\ (n-2)\,B\,x'^{n-3} \\ \cdot\;\cdot \\ T \\ \end{array}\right| u + \left.\begin{array}{r} n\dfrac{n-1}{1.\,2}x'^{n-2} \\ (n-1)\,\dfrac{n-2}{1.\,2}A\,x'^{n-3} \\ (n-2)\,\dfrac{n-3}{1.\,2}B\,x'^{n-4} \\ \cdot\;\;\cdot\;\;\cdot\;\;\cdot \end{array}\right| u^{2} +$$

$$\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\; u^{n} = 0. \quad (2)$$

Observing the manner in which the different coefficients of u are formed, the following remarkable law will be discovered:

1°. The coefficient of u° is simply what the first equation becomes when x' is substituted for x.

Let us designate this coefficient by X.

2°. The coefficient of u is formed by means of the preceding or X, by multiplying each of the terms of X by the exponent of x' in that term, and diminishing this exponent by unity.

Denote this coefficient by Y.

3°. The coefficient of u^2 is formed from Y, by multiplying each term of Y by the exponent of x' in that term, dividing the product by 2, and diminishing each exponent by unity.

Let $\frac{Z}{2}$ be this coefficient. It is evident that Z is formed in the same manner from Y that Y is formed from X. In general the coefficient of any power of u is formed from the preceding coefficient in the following manner, viz.:

By taking each term of the preceding coefficient in succession, multiplying it by the exponent of x', *dividing by the number which marks the place of the coefficient, and diminishing the exponent of* x' *by unity.*

The preceding development will be represented generally by

$$X + Yu + \frac{Z}{2}u^2 + \frac{V}{2.3}u^3 + \&c.$$

The polynomials, Y, Z, &c., are called *derived polynomials of* X, since Z is derived in the same manner from Y, that Y is

from X. Y is called the *first*, Z the *second*, V the *third*, derived polynomial of X, and so on.

Ex. Let $X = x^3 - 6x^2 + 11x - 6 = 0$. To find the derived polynomials.

$$\text{Ans. } Y = 3x^2 - 12x + 11$$
$$Z = 6x - 12$$
$$V = 6.$$

238. Returning to our purpose, let it now be proposed to find an equation, the roots of which shall be greater by unity than those of the equation

$$4x^3 - 5x^2 + 7x - 9 = 0.$$

Let us put $x = u - 1$, from which we have $u = x + 1$. From the preceding development the transformed equation will be

$$X + Yu + \frac{Z'}{2}u^2 + \frac{V}{2.3}u^3,$$

in which $x' = -1$. To find the coefficients, we have therefore

$$X = 4(-1)^3 - 5(-1)^2 + 7(-1) - 9 = -25$$
$$Y = 12(-1)^2 - 10(-1) + 7 \qquad = \quad 29$$
$$\frac{Z}{2} = 12(-1)^1 - 5 \qquad = -17$$
$$\frac{V}{2.3} = 4 \qquad = \quad 4$$

The transformed required will be

$$4u^3 - 17u^2 + 29u - 25 = 0.$$

Ex. 2. Transform the equation $x^4 - 4x^3 - 8x + 32 = 0$ into another, whose roots shall be 2 less.

$$\text{Ans. } x^4 + 4x^3 - 24x = 0.$$

239. The process above being still laborious, another more simple is to be sought.

In order to this, we resume the general equation, and also equation (2) art. 237, denoting the coefficients of this last by $A', B' \;.\;.\;.\;.\; T', U'$, respectively, thus,

$$x^n + Ax^{n-1} + Bx^{n-2} + \;.\;.\;.\;.\;.\; Tx + U = 0, \quad (1)$$
$$u^n + A'u^{n-1} + B'u^{n-2} + \;.\;.\;.\;.\;.\; T'u + U' = 0. \quad (2)$$

Substituting in this last for u its value $x - x'$, we obtain

$(x-x')^n + A'(x-x')^{n-1} + B'(x-x')^{n-2} + \ldots . T'(x-x') + U' = 0,$ (3)

which when developed must, it is evident, be identical with equation (1); since the second is formed from the first by the substitution of $u+x'$ for x, and the third is formed from the second by substituting $x-x'$ for u, by which we necessarily return to (1), the original equation. We have, therefore,

$(x-x')^n + A'(x-x')^{n-1} + \ldots \; T'(x-x') + U' = x^n + Ax^{n-1} + \ldots \; Tx + U.$ (4)

Now, if we divide the first member of this equation by $x-x'$, every term will be divisible by it, except the last, which will be the remainder after the division. And since the two members of this equation are identically the same, we shall obtain the same quotient and the same remainder by dividing the second member by $x-x'$. But U' is the last or absolute term of the transformed equation. It follows, therefore, that if we divide the proposed equation by $x-x'$, the remainder will be equal to the last or absolute term of the transformed equation.

Again, let the quotient arising from the division of the first member of (4) by $x-x'$ be represented by

$(x-x')^{n-1} + A'(x-x')^{n-2} + \ldots \; R'(x-x') + T'.$

If we now divide this quotient by $x-x'$, every term will be divisible by it, except the last, or T', the coefficient of the last term but one of the transformed. And, since the remainder must be the same when the quotient of the second member of (4) by $x-x'$ is also divided by $x-x'$, we shall obtain, it is evident, the coefficient of the last term but one of the transformed equation by dividing this last quotient also by $x-x'$. And it is easy to see that thus, by successive divisions, all the coefficients of the transformed equation may be obtained.

Thus, resuming the equation already transformed, viz. : $4x^3 - 5x^2 + 7x - 9 = 0$, and dividing by $x+1$, we obtain a quotient $4x^2 - 9x + 16$, and a remainder -25. Dividing next this quotient by $x+1$, we obtain a new quotient $4x - 13$, and

a remainder 29. Dividing next this last quotient by $x+1$, the operation terminates, and we have a remainder -17. The transformed will be, therefore, as before,

$$4u^3-17u^2+29u-25=0.$$

We have, then, the following rule by which to obtain the transformed equation:

Let x' be the quantity by which the roots of the equation are required to be increased or diminished. Divide the equation by $x-x'$, or $x+x'$, as the case may require, and the quotient thus obtained by the same quantity, and so on until the division terminates. *The coefficient of the first term, or that containing the highest power of the transformed, will be the same with the coefficient of the first term of the proposed equation; and the successive coefficients will be the remainders arising from the successive divisions taken in a reverse order, the first remainder being the last or absolute term of the transformed equation.*

The successive divisions, which are tedious by the common method, are performed in a very concise and elegant manner by synthetic division.

Thus, in the last example,

$$\begin{array}{r|rrrr}
1 & 4 & -5 & +7 & -9 \\
-1 & & -4 & +9 & -16 \\
\cline{2-5}
 & & -9 & +16 & -25 \\
 & & -4 & +13 & \\
\cline{2-4}
 & & -13 & +29 & \\
 & & -4 & & \\
\cline{2-3}
 & & -17 & &
\end{array}$$

In which the remainders, as before, are -25, 29 and -17.

Examples.

1. Transform the equation $5x^4+28x^3+51x^2+32x-1=0$ into another, having its roots greater by 2 than those of the proposed equation.

Operation.

$$\begin{array}{r|l}
1 & 5 + 28 + 51 + 32 - 1 \\
-2 & \quad - 10 - 36 - 30 - 4 \\
\hline
 & 5 + 18 + 15 + \ \ 2 - 5 \\
 & \quad - 10 - 16 + \ \ 2 \\
\cline{2-2}
 & 5 + \ \ 8 - \ \ 1 + \ \ 4 \\
 & \quad - 10 + \ \ 4 \\
\cline{2-2}
 & 5 - \ \ 2 + \ \ 3 \\
 & \quad - 10 \\
\cline{2-2}
 & \quad - 12
\end{array}$$

Ans. $5x^4 - 12x^3 + 3x^2 + 4x - 5 = 0.$

2. Find an equation whose roots shall be less by $\frac{1}{3}$ than those of the equation $3x^4 - 13x^3 + 7x^2 - 8x - 9 = 0$.

Placing the divisor at the right, as in ordinary division, and omitting all the figures in the first column except the first, the calculations may be more conveniently disposed, thus:

$$\begin{array}{rrrrr|r}
3 & -13 & +7 & -8 & -9 & 1 \\
 & 1 & -4 & 1 & -2\frac{1}{3} & \frac{1}{3} \\
\cline{2-6}
 & -12 & 3 & -7 & -11\frac{1}{3} & \\
 & 1 & -3\frac{2}{3} & -\frac{2}{9} & & \\
\cline{2-4}
 & -11 & -\frac{2}{3} & -7\frac{2}{9} & & \\
 & 1 & -3\frac{1}{3} & & & \\
\cline{2-3}
 & -10 & -4 & & & \\
 & 1 & & & & \\
\cline{2-2}
 & -9 & & & &
\end{array}$$

Ans. $3x^4 - 9x^3 - 4x^2 - \dfrac{65}{9}x - \dfrac{102}{9} = 0.$

3. Transform the equation $x^3 - 12x - 28 = 0$ into another, whose roots shall be less by 4.

In the use of synthetic division the missing term in this example must be supplied by 0 coefficient.

Ans. $x^3 + 12x^2 + 36x - 12 = 0.$

4. Transform the equation $3x^4 - 4x^3 + 7x^2 + 8x - 12 = 0$ into another, whose roots shall be less by 3.

Ans. $3x^4 + 32x^3 + 133x^2 + 266x + 210 = 0.$

5. Find the equation whose roots shall be greater by 2 than those of the equation $x^5+10x^4+42x^3+86x^2+70x+12=0$. Ans. $x^5+2x^3-6x^2-10x+8=0$.

6. Find the equation whose roots shall be less by $\frac{1}{2}$ than those of the equation $2x^4-6x^3+4x^2-2x+1=0$.

$$\text{Ans. } 2x^4-2x^3-2x^2-\frac{3}{2}x+\frac{3}{8}=0.$$

240. The transformation above is the only one for which we have immediate occasion. There are others, which are sometimes useful, which we will here explain.

1. From what has been done we may readily transform an equation into another, deprived of its second term.

Resuming the general equation $x^n+Ax^{n-1}+\ldots Tx+U=0$, and putting $x=u+x'$, developing and arranging with reference to the descending powers of u, we have

$$u^n+(nx'+A)u^{n-1}\ \ldots\ldots =0.$$

In order that the second term of this last may disappear, its coefficient must be equal to 0. We shall have then the relation

$$nx'+A=0;\ \text{whence } x'=-\frac{A}{n}.$$

Hence, to make the second term of an equation disappear, 1°. *Divide the coefficient of the second term by the highest power of the unknown quantity.* 2°. *Transform the equation into another whose roots shall be less or greater by the quotient thus obtained, according as the sign of the second term is negative or positive.*

EXAMPLES.

Deprive the following equations of their second terms:

1. $x^3-6x^2+4x-7=0$. Ans. $x^3-8x-15=0$.
2. $x^4-8x^3+16x^2+7x-12=0$.
   Ans. $x^4-8x^2+7x+18=0$.
3. x^3-6x^2+8x-2. Ans. $x^3-4x-2=0$.
4. $x^5+15x^4+12x^3-20x^2+14x-25=0$.
   Ans. $x^5-78x^3+412x^2-757x+401=0$.

2. An equation may be transformed into another, the roots of which shall be any multiple of those of the proposed equation.

In the general equation $x^n + A x^{n-1} + \dots T x + U = 0$, let $x = \frac{x}{m}$. Substituting $\frac{x}{m}$ for x, and clearing of fractions, we obtain

$$x^n + A m x^{n-1} + B m^2 x^{n-2} \dots \dots T m^{n-1} x + U m^n = 0,$$

the roots of which are m times greater than those of the proposed. The required equation is, therefore, found by multiplying the second coefficient of the proposed by m, the third by m^2, and so on, m representing the number of times the required roots are to exceed those of the proposed.

Ex. Find an equation whose roots shall be three times larger than those of the equation $x^3 - 6x^2 + 8x - 9 = 0$.

Ans. $x^3 - 18x^2 + 72x - 243 = 0$.

By means of this principle we may transform an equation with fractional coefficients into another, whose coefficients shall be integral. In order to this, we have merely to transform the proposed equation into another, the roots of which shall be equal to those of the proposed multiplied by the least common multiple of the denominators of the fractions.

Ex. 1. Transform the equation $x^3 + \frac{1}{3}x^2 - \frac{1}{4}x + 2 = 0$ into another, the coefficients of which shall be integral.

Ans. $x^3 + 4x^2 - 36x + 3456 = 0$.

In this example the number 6, when raised to the square, will, it is evident, be divisible by 4. The fractional coefficients may, therefore, be removed, and a more simple result obtained, if we multiply by the successive powers of 6. It will be easy to apply the like simplification to other cases.

Ex. 2. Transform the equation $x^3 - \frac{7}{3}x^2 + \frac{11}{36}x - \frac{25}{72} = 0$, into another, the coefficients of which shall be integral.

Ans. $x^3 - 14x^2 + 11x - 75 = 0$.

3. In like manner an equation may be transformed into another, the roots of which shall be the reciprocals of the proposed equation.

In order to this, substituting $\frac{1}{x}$ for x in the general equation, clearing of fractions, and reversing the order of the terms, it becomes

$$U x^n + T x^{n-1} + S x^{n-2} + \ldots \ldots A x + 1 = 0,$$

the roots of which are reciprocals of those of the proposed equation.

The transformation is therefore effected by simply changing the order of the coefficients of the proposed equation.

If the coefficients of the proposed equation are the same, whether taken in the reverse or direct order, the transformed equation, it is evident, will be identical with the given equation, and will furnish the same series of roots.

Equations of this description are called *recurring* equations, or, from the form of the roots, *reciprocal* equations.

4. From what has been done, an equation, it is evident, may be transformed into another, whose roots shall be greater or less than the reciprocals of the proposed equation. In order to this, we reverse the order of the coefficients, and then apply the process of art. 237.

Ex. 1. Transform the equation $x^3 - 7x + 7$ into another, the roots of which shall be less by 1 than the reciprocals of those of the proposed. Ans $7x^3 + 14x^2 + 7x + 1 = 0$.

Ex. 2. Find the equation whose roots shall be the reciprocals of those of $3x^4 - 13x^3 + 7x^2 - 8x - 9 = 0$, increased by 2.

Ans. $-9x^4 + 64x^3 - 161x^2 + 151x - 23 = 0$.

LIMITS OF THE ROOTS.

241. The methods for resolving numerical equations of any degree consist, in general, in substituting particular numbers for x in the equations, in order to see if these numbers will verify them; or for the purpose of determining the initial figures

of the roots. It is important, therefore, as a preliminary step, to determine the limits between which the roots of an equation are to be sought.

1. A number numerically greater than the greatest positive root of an equation is called a *Superior Limit to the positive roots.* A number numerically greater than the greatest negative root, abstraction being made of the sign, is called a *Superior Limit* to the negative roots.

The extreme limits to the positive roots, it is evident, will be 0 and $+\infty$, and those of the negative roots 0 and $-\infty$, the sign ∞, being used to denote infinity. In practice, much narrower limits than these will be required.

Since the largest of the positive roots when substituted for x, reduces the equation to 0, it follows that a number greater than this, when substituted in like manner for x, will give a positive result. If, therefore, *a number substituted for* x *in an equation gives a positive result, then this number will be a superior limit to the positive roots.* (No criterion: for there may be roots greater than that value of x which makes fx positive.

1. It may be shown that in any equation the greatest *negative coefficient increased by unity is a superior limit to the positive roots.* A much nearer limit may, however, be found.

2. Let us take the general equation,

$$x^n + A x^{n-1} + \ldots - D x^{n-r} \ldots\ldots - P x^{n-s} \ldots\ldots U = 0,$$

and let $-D x^{n-r}$ be the first negative term, and $-P$ the greatest negative coefficient; then, if all the terms after the first negative term are negative, the sum of these negative terms, it is evident, must be equal to the preceding positive terms. And any value, which, substituted for x, will make the sum of the positive terms greater than the sum of the negative terms, will be a superior limit. And, P being the greatest negative coefficient, for a still stronger reason, any number will be a superior limit, which substituted for x gives

$$x^n > P\,(x^{n-r} + x^{n-r-1} + \ldots\ldots x + 1), \quad (1)$$

or, since the right hand factor of this inequality is a progression by quotient, art. 176,

$$x^n > P\left(\frac{x^{n-r+1}-1}{x-1}\right) \qquad (2)$$

But this inequality will be satisfied if we have

$$x^n > P\left(\frac{x^{n-r+1}}{x-1}\right)$$

or, reducing $\quad (x-1)\,x^{r-1} > P; \qquad (3)$

but $x-1$ is less than x, and by consequence $(x-1)\,(x-1)^{r-1}$, or, $(x-1)^r$ is less than $(x-1)\,x^{r-1}$; the inequality (3) will, therefore, be satisfied if we have

$$(x-1)^r =, \text{ or } > P, \text{ or } x =, \text{ or } > P^{\frac{1}{r}} + 1.$$

Thus, to obtain a superior limit of the positive roots, *we increase by unity the root of the greatest negative coefficient, whose index is the number of terms which precede the first negative term.*

EXAMPLES.

Find superior limits to the roots of the following equations:

1. $x^5 + 7x^4 - 12x^3 - 49x^2 + 52x - 13 = 0.$

Ans. $P^{\frac{1}{r}} + 1 = (49)^{\frac{1}{2}} + 1 = 8.$

2. $x^4 - 5x^3 + 37x^2 - 3x + 39 = 0.$ Ans. 6.

3. $3x^3 - 2x^2 - 11x + 4 = 0.$ Divide the equation firs by 3, then applying the rule the limit will be 5.

4. $x^4 + 11x^2 - 25x - 67 = 0.$ The greatest negative co efficient, is that of x^0, or 67, and the missing term being counted which is necessary, $r = 3$; whence $P^{\frac{1}{r}} + 1 = (67)^{\frac{1}{3}} + 1 = 6$

5. $x^4 + 11x^2 - 25x - 61 = 0.$ Ans. 5.

242. A number numerically less than the least positive root of an equation is called *an inferior limit to the positive roots.*

In the preceding examples we have regarded 0 as the inferior limit of the positive roots. Thus the positive roots of the first example are all comprised between 0 and 8. A nearer inferior 'imit may, however, be found.

Let $X = 0$ be any equation, and in this equation let us put

$x = \frac{1}{y}$. We shall obtain, it is evident, a derived equation $Y = 0$ in which the greatest value of y will correspond to the least value of x. If then we find the superior limit L to the roots of the equation $Y = 0$, the reciprocal of this, or $\frac{1}{L}$, will be the inferior limit to the roots of the proposed equation, or $X = 0$.

Thus, let it be proposed to find the inferior limit of the positive roots of the equation $x^3 - 42x^2 + 441x - 49 = 0$.

Putting $x = \frac{1}{y}$, we have for the transformed equation,

$$y^3 - 9y^2 + \frac{6}{7}y - \frac{1}{49} = 0,$$

which gives, by the rule, 10 for the superior limit of its positive roots. Thus the inferior limit to the positive roots of the proposed, will be $\frac{1}{10}$.

243. The particular form of the equation may sometimes suggest artifices, by means of which closer limits may be obtained than those given by the preceding rules.

Thus, the equation $x^4 + 11x^2 - 25x - 61 = 0$, may be put under the form

$$x(x^3 - 25) + 11\left(x^2 - \frac{61}{11}\right)$$

in which it is evident that $x = 3$, or any number greater than 3, will give a positive result. We shall have 3, therefore, for the superior limit to the positive roots, which is much nearer than 5, the limit obtained by the rule.

The equation $x^4 - 5x^3 + 37x^2 - 3x + 39 = 0$, may be put under the form

$$x^3(x - 5) + 37x\left(x - \frac{3}{37}\right) + 39 = 0;$$

and the equation $x^5 + 7x^4 - 12x^3 - 49x^2 + 52x - 13 = 0$ under the form

$$x^2(x^3 - 49) + 7x^3\left(x - \frac{12}{7}\right) + 52\left(x - \frac{1}{4}\right) = 0.$$

It is evident that 5, and any number greater than 5, substituted for x in the first of these equations, and 4, or any number greater than 4, substituted for x in the second, will give positive results. We have, therefore, 5 and 4 respectively for the limits, instead of 6 and 8 obtained by the rule.

The artifice consists in decomposing the equation into parts, each of which is composed of two factors, the first a positive monomial, and the other a binomial in x, the second term of which is negative, and then determining x in such a manner, that all the factors within parentheses shall be positive.

244. It remains to find the superior and inferior limits to the negative roots. In order to this, we transform the proposed equation into another whose roots shall be the same as the proposed with contrary signs. The limits of the positive roots of this equation, taken with a contrary sign, will be the limits to the negative roots of the proposed equation.

Thus, let it be proposed to find the limits to the negative root of the equation $x^3 - 7x + 7 = 0$. Putting $x = -x$, or, which is the same thing, changing the signs of the alternate terms, the missing term being supplied, we have

$$x^3 - 7x - 7 = 0,$$

the limits to which are 4 and 3 respectively. The negative root of the proposed will, therefore, lie between -4 and -3.

245. In the preceding numbers we have found the limits between which the roots *all* lie. When the roots of the proposed are incommensurable we shall wish to find limits between which the *individual* roots are situated, in order to determine more readily the initial figures of the roots.

Let a, b, c, &c., be the real roots of an equation in the order of their magnitude, so that we have $a > b$, $b > c$, &c.; and let a', b', c', be a series of numbers, such that a' is greater than a, b' a number comprised between a and b, so that we have $b' < a$, $b' > b$, and so on.

The original equation, it is evident, will be

$$(x - a)\,(x - b)\,(x - c)\;.\;.\;.\;.\;. = 0.$$

If now, in this equation, we substitute a' for x, it becomes

$$(a'-a)\ (a'-b)\ (a'-c)\ .\ .\ .\ .=0\,;$$

and since a' is greater than a, b, c, &c., the factors will each be positive, and hence their product will be *positive.*

Again, let b' be substituted for x, the equation becomes

$$(b'-a)\ (b'-b)\ (b'-c)\ .\ .\ .\ .\ .=0\,;$$

here, since b' is less than a, but greater than b, c, &c., the first factor will be negative and the rest positive; the product, therefore, will be *negative.*

If, again, c' be substituted for x, the equation becomes

$$(c'-a)\ (c'-b)\ (c'-c)\ .\ .\ .\ .\ .=0\,;$$

here, since c' is less than a and b, but greater than c, &c., the first two factors will be negative and the rest positive; the product will, therefore, be *positive.*

Hence, 1°, If a quantity, *greater than the greatest real root,* be substituted for x, the result will be *positive.*

2°. If quantities *intermediate* between the roots, beginning with the greatest, be substituted, the results will be *alternately negative* and *positive.*

Ex. 1. The roots of the equation $x^3-13\,x+12=0$, are 3, 1, and -4. Substitute 4, 2, 0, and -5 for x, and observe the signs of the results.

Ex. 2. Make a like substitution in the equation $x^3-5\,x^2+2\,x+8=0$, the roots of which are 4, 2, and -1.

From the preceding principles it follows,

1°. If two numbers be successively substituted for x in any equation, and give results with *different* signs, then between these numbers there must be *one, three, five,* or some *odd* number of roots.

2°. But if the numbers substituted for x give the *same* sign, then between these numbers there will be either *no* root, or there will be *two, four,* or some *even* number of roots.

3°. If any quantity q and every quantity greater than q

renders the result *positive*, then q *is greater than the greatest root* of the equation, and will be a *superior limit* of the roots.

4°. Hence, if the *signs of the alternate terms are changed*, and if p and every quantity greater than p renders the result *positive*, then $-p$ is *less than the least root*, and will be an *inferior* limit.

5°. If the degree of the equation be *even*, the substitution of a number less than the least root will give a *positive* result. But if the degree be *odd*, the result will be *negative*.

246. The substitution of the natural series, 0, 1, 2, 3, &c., taken negatively as well as positively, will enable us to discover the position, and determine, in general, the initial figure of the real roots.

Ex. 1. Let it be required to find one of the roots of the equation $x^3 - 4x^2 - 6x + 8 - 0$.

Substituting 0 for x, we have 8 for the result. Substituting next 1, the result will be -1. There will be a root, therefore, between 0 and 1, and very near 1. Try next .9. Putting .9 for x, the result is .089. There is, therefore, a root between .9 and 1. We shall have then .9 for the initial figure of the root.

Ex. 2. Find the first figure of one of the roots of the equation $x^4 + 3x^3 + 2x^2 + 6x - 148 = 0$.

Ans. The substitution of 2 gives a negative, and of 3 a positive result. There is, therefore, a root between 2 and 3. Hence 2 is the first figure of the root.

Ex. 3. Find the first figure of one of the roots of the equation $x^3 + x^2 + x - 100 = 0$. Ans 4.

Ex. 4. Find the first figure of one of the roots of the equation $x^3 + 1.5x^2 + .3x - 46 = 0$. Ans. 3.

Ex. 5. Find the first figure of one of the roots of the equation $x^3 - 12x + 8 = 0$. Ans. .6.

LIMITING EQUATION. EQUAL ROOTS.

247. An equation, the roots of which are intermediate between those of a proposed equation, will have for its roots limits to those of the proposed equation, and is, therefore, called the *separating* or *limiting* equation.

Let a, b, c, &c., taken in the order of their magnitude, be the roots of the equation

$$x^n - \mathrm{A}\,x^{n-1} + \mathrm{B}\,x^{n-2} + \ldots\ldots \mathrm{T}\,x + \mathrm{U} = 0; \quad (1)$$

it is required to find an equation the roots of which shall lie between or separate those of the proposed equation.

Diminishing the roots of the proposed by x', we put $x = u + x'$, and developing, as in art. 237, it becomes

$$u^n + \mathrm{A}'\,u^{n-1} + \ldots\ldots \mathrm{T}'\,u + \mathrm{U}' = 0,$$

in which the coefficient of the last term, U′, will be

$$x'^n + \mathrm{A}\,x'^{n-1} + \mathrm{B}\,x'^{n-2} + \ldots \mathrm{T}\,x' + \mathrm{U};$$

and the coefficient T′ of the term before the last will be

$$n\,x'^{n-1} + (n-1)\,\mathrm{A}\,x'^{n-2} + (n-2)\,\mathrm{B}\,x'^{n-3} + \ldots\ldots \mathrm{T}. \quad (2)$$

But a, b, c, &c., being roots of equation (1), the roots of the transformed will be $(a - x')$, $(b - x')$, $(c - x')$, &c. Hence, by art. 229, the coefficient of the last term but one of the transformed will be the sum of the products of every $n - 1$ of these roots with their signs changed. Thus this coefficient will be

$$\left.\begin{array}{l} (x'-b)\,(x'-c)\,(x'-d) \text{ to } n-1 \text{ factors,} \\ +\,(x'-a)\,(x'-c)\,(x'-d) \quad \text{“} \quad\quad \text{“} \\ +\,(x'-a)\,(x'-b)\,(x'-d) \quad \text{“} \quad\quad \text{“} \\ +\,(x'-a)\,(x'-b)\,(x'-c) \quad \text{“} \quad\quad \text{“} \\ +\,\&\text{c.} \end{array}\right\} \quad (3)$$

all the factors save one occurring, of course, in each term.

But the expressions (2) and (3) are equal, since they are but different expressions for the same thing, viz., the last coefficient but one of the transformed equation. Hence, whatever changes are produced by substitution in one of these expressions, the

same changes will be produced by a like substitution in the other.

If we now substitute a for x' in each term of (3), all the terms, except the first, will vanish, and the coefficient will be reduced to $(a-b)\ (a-c)\ (a-d)\ \ldots$ in which the signs of the factors are each positive, since a is greater than b, c, d, &c. Substituting b, in like manner, all the terms vanish except the second, in which the sign of the first factor will be negative, and those of the rest positive. For the substitution of a, b, c, &c., successively, we shall have the following results.

$$\begin{array}{lllll}
\text{1st, } a, & (a-b) & (a-c) & (a-d) & \ldots = + \,.\, + \,.\, + = + \\
\text{2d, } b, & (b-a) & (b-c) & (b-d) & \ldots = - \,.\, + \,.\, + = - \\
\text{3d, } c, & (c-a) & (c-b) & (c-d) & \ldots = - \,.\, - \,.\, + = + \\
\text{4th, } d, & (d-a) & (d-b) & (d-c) & \ldots = - \,.\, - \,.\, - = - \\
 & \text{\&c., \&c.} & & &
\end{array}$$

and by consequence the same changes of sign will result from a like substitution in (2).

If then we put $(2) = 0$, and write x for x', it becomes

$$n\,x^{n-1} + (n-1)\,A\,x^{n-2} + B\,(n-2)\,x^{n-3} + \ldots\ldots T = 0. \quad (4)$$

And since, from what has been done, the substitution of a, b, c &c., in this last gives results alternately positive and negative its roots will be intermediate between these quantities, that is, between the roots of equation (1).

Equation (4) will, therefore, be the limiting equation to equation (1); and since the former is the derivative of the latter, in general, *the derivative of an equation will be its limiting equation.*

Ex. 1. What is the limiting equation to $x^4 - 7x^3 + 5x^2 + 31x - 30 = 0$? Ans. $4x^3 - 21x^2 + 10x + 31 = 0$.

248. The roots of equation (1) in the preceding article, being represented by a, b, c, &c., respectively, if b', c', d', &c., in like manner represent the roots of equation (4), the roots of the two equations arranged in the order of their magnitude will stand thus:

$$a,\ b',\ b,\ c',\ c,\ d',\ \text{\&c.}$$

Here, if the difference between a and b is zero, the difference between a and b' must also be zero; that is, if a is equal to b, it must also be equal to b'; hence the factor $x-a$ will be found both in the proposed and its limiting equation. The two equations will, therefore, have a common measure $x-a$.

Again, if b and c are each equal to a, then b', c', will also be each equal to a, and the two equations will have a common measure $(x-a)^2$, and so on.

Conversely, if a proposed equation and its derivative have a common measure $x-a$, the proposed will have two roots, each equal to a. If they have a common measure, $(x-a)^2$, the proposed will have three roots, each equal to a, and so on.

To determine, therefore, whether an equation has equal roots, *we form the derivative or limiting equation, and then seek the greatest common divisor of the two equations.*

The factors of this common divisor being determined, it is evident they must enter each once more into the proposed, and thus the number and value of the equal roots will be determined.

Suppose the greatest common divisor of the proposed and its derivative to be $(x-a)^3(x-b)^2(x-c)$. The proposed will have 4 roots equal to a, 3 equal to b, and 2 equal to c.

Ex. 1. Find the equal roots of the equation $3x^5-10x^3+15x+8=0$.

The derivative is $15x^4-30x^2+15=0$. And the greatest common divisor is x^2+2x+1, or, $(x+1)^2$. The proposed, therefore, has 2 roots equal each to -1.

Ex. 2. Find the equal roots of the equation $x^4-14x^3+61x^2-84x+36=0$.

The greatest common divisor between the proposed and its derivative is $x^2-7x+6=(x-6)(x-1)$.

Ans. Two roots equal to 6 and two equal to 1.

Ex. 3. Find the equal roots of the equation $x^4-2x^2+1=0$. Ans. Two equal to 1, and two equal to -1.

Ex. 4. Find the equal roots of the equation $x^3-2x^2-4x+8=0$. Ans. Two equal to 2.

Ex. 5. Find the equal roots of the equation $x^4-6x^3+8x^2+6x-9=0$. Ans. Two equal to 3.

IMAGINARY AND REAL ROOTS. STURM'S THEOREM.

249. In the search for the roots of an equation, it is of the first importance to determine, at the outset, the precise number of real roots the proposed equation contains, in order that the substitutions required in the solution may be restricted within the narrowest possible limits. To separate the real and imaginary roots of a proposed equation so as to determine the exact number of each, is a problem of great and acknowledged difficulty. It entirely baffled the skill of mathematicians, until in 1829 it was completely solved by the beautiful Theorem of Sturm, which we shall now explain.

Let $V=x^n+Ax^{n-1}+Bx^{n-2}+\ldots Tx+U=0$,

be an equation of the nth degree with no equal roots, and $V_1=0$ be its derivative or limiting equation.

We now apply to V and V_1 the process for finding their greatest common divisor until a remainder is obtained independent of x, observing, however, to change the signs of each remainder as we proceed.

Let the series of remainders, with their signs changed, be represented by $V_2, V_3, V_4 \ldots V_n$, V_n being the last remainder, or that which is independent of x.

Let p and q be any numbers taken at pleasure, of which p is the greater. Let p be substituted in the place of x in the functions $V, V_1, V_2 \ldots\ldots V_n$, and write in order in one line the signs of the results. Substitute next q in the same manner, and write in order the signs of the results. *The difference in the number of variations between the two rows of signs resulting from these substitutions, will be equal to the number of real roots of the equation $V=0$, comprised between the numbers* p *and* q.

This is the Theorem of Sturm, which we now proceed to demonstrate.

In order to this, let Q, Q_1, Q_2, &c., represent the quotients obtained in the successive divisions of V by V_1, V_1 by V_2, and so on. From the manner in which these functions are derived, we shall have this series of equations:

$$\begin{aligned} V &= Q\ V_1 - V_2 \\ V_1 &= Q_1 V_2 - V_3 \\ V_2 &= Q_2 V_3 - V_4 \\ V_3 &= Q_3 V_4 - V_5 \\ &\cdots\cdots \\ V_{n-2} &= Q_{n-2} V_{n-1} - V_n \end{aligned}$$

This being premised, we remark

1°. *No two consecutive functions can become* 0, *or vanish, for the same value of* x.

For if two consecutive functions V_2, V_3, for example, can at the same time become equal to 0, then we shall have $V_4 = 0$; and V_3, V_4 being each equal to 0, then V_5 will be equal to 0, and so on, until finally V_n or the last remainder will be equal to 0, which is impossible, since this remainder is independent of x, and cannot be affected by any change in the value of x.

2°. *If one of the functions*, V_3, *for example, becomes* 0 *for a particular value of* x, *then the adjacent functions between which it is placed, have for that value contrary signs.*

For we have $V_2 = Q_2 V_3 - V_4$;
hence, if $V_3 = 0$, $V_2 = -V_4$.

Thus the adjacent functions have in this case contrary signs.

Let now p be greater than the greatest root, and negative roots being regarded as less than corresponding positive ones, let q be less than the least root of the equations

$$V = 0,\ V_1 = 0,\ V_2 = 0\ \ldots\ldots\ V_{n-2} = 0,$$

and let q increasing by insensible degrees until it reaches the value of p, be substituted successively for x in these equations. We remark again,

1°. So long as q remains less than the least root of the equa-

tions, no change will be produced in the signs by its substitution.

2°. But when q in the process of its increase becomes equal to the least root of the equations, in V_3, for example, then for this value V_3 vanishes. But there will be no change in the number of variations of the signs produced by this circumstance.

Indeed, q being still less than the least root of V_2 and V_4, the signs of these will not change by the substitution which causes V_3 to vanish. And since when V_3 vanishes, V_2 and V_4 have necessarily opposite signs, the signs of the three consecutive functions must before have been, either

$$\left.\begin{matrix} V_2, & V_3, & V_4 \\ + & \mp & - \end{matrix}\right\} \text{ or } \left\{\begin{matrix} V_2, & V_3, & V_4 \\ - & \pm & + \end{matrix}\right.$$

which give each, whichever of the double signs is the true one, one permanence and one variation.

Now, when V_3 vanishes, the signs become

$$\left.\begin{matrix} V_2, & V_3, & V_4 \\ + & 0 & - \end{matrix}\right\} \text{ or } \left\{\begin{matrix} V_2, & V_3, & V_4 \\ - & 0 & + \end{matrix}\right.$$

in each of which there is still one variation.

If q now becomes greater than the least root of V_3, but is still less than the roots of V_2, V_4, the signs of V_2, V_4 will remain as they were before; but the sign of V_3 will change. The three consecutive signs will then be

$$+ \pm -, \text{ or } - \mp +,$$

still exhibiting one permanence and one variation.

The same reasoning obviously applies to any of the intermediate functions between V and V_n.

The function V_n, being the last remainder and independent of x, can undergo no change of sign, whatever value is substituted for x. It follows, therefore, *that there can be no change in the number of variations of the signs, unless it arise from a change of sign in the primitive function.*

3°. Of the two equations $V = 0$, $V_1 = 0$, one will necessarily be of an even degree and the other odd. If, therefore, a number less than their least roots be substituted in them, the results

(art. 245) will be of *different signs.* Let us then suppose next that the value of q in its increase has now become greater than the least root of $V = 0$, but is still less than the least root of $V_1 = 0$, the roots of the latter (art. 247) being necessarily greater than the least root of the former. When q passes the least root of $V = 0$, there will be a change of sign in this equation; thus the results of the substitution in the equations $V = 0$, $V_1 = 0$ will now have *the same sign.* That is, the results of the substitution in these two equations, which before exhibited a variation, will, in its stead, exhibit a permanence. And by consequence the whole number of variations is in this case diminished by unity.

If q goes on to increase until it has passed the least root of $V_1 = 0$, this function will change sign, so that V and V_1 will give different signs, but will again have the same sign when the second root of $V = 0$ is passed. Thus there will be no change in the number of variations until the second root of $V = 0$ is passed, when the number of variations will again be diminished by unity.

In like manner it may be shown that when q in its progress towards p passes successively each of the remaining roots of the primitive function $V = 0$, the number of variations in each case will be diminished by unity.

It follows, therefore, since no change in the number of variations is made when any of the functions except the primitive are reduced to zero, that *the difference in the number of variations, when* p *and* q *are successively substituted in the functions* V, V_1, V_2, *&c., will always be equal to the number of real roots comprised between* p *and* q.

This is the proposition which was to be demonstrated. We will illustrate by an example.

For this purpose let us form the equation whose roots shall be 1, 2, 3, and 4. It will be $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$.

The functions formed according to the rule, and their roots, when the functions are made equal to 0, will be as follows:

Functions.	*Roots.*
$V = x^4 - 10x^3 + 35x^2 - 50x + 24$	1, 2, 3, 4
$V_1 = 4x^3 - 30x^2 + 70x - 50$	1.3, 2.5, 3.6 nearly
$V_2 = 5x^2 - 25x + 29$	1.8, 3.1 "
$V_3 = 2x - 5$	2.5
$V_4 = 9$	

Since the least root of the functions is 1, we will begin the substitutions with a number less than this .8, for example.

		V	V_1	V_2	V_3	V_4	
$x = .8$	gives	+	−	+	−	+	4 variations.
.9	"	+	−	+	−	+	4 "
1	"	0	−	+	−	+	
1.1	"	−	−	+	−	+	3 "
1.9	"	−	+	−	−	+	3 "
2	"	0	+	−	−	+	
2.1	"	+	+	−	−	+	2 "
2.5	"	+	0	−	0	+	2 "
2.9	"	+	−	−	+	+	2 "
3	"	0	−	−	+	+	
3.1	"	−	−	−	+	+	1 variation.
3.9	"	−	+	+	+	+	1 "
4	"	0	+	+	+	+	
4.1	"	+	+	+	+	+	no variation.

From inspection of this table, we see

1°. So long as the value substituted for x is less than the least root of the functions there is no change in the signs.

2°. When in the process of the substitution V_1, V_3 vanish, the adjacent functions are of opposite signs, and no change takes place in the number of the variations.

3°. That whatever changes of sign take place in the secondary functions V_1, &c., the number of variations is not affected by these changes.

4°. That in every case when the primitive function changes sign, there is a loss of one variation and of one only.

5°. The roots of the equation being comprised within the

limits of the values assigned to x, the number of variations lost is precisely equal to the number of the real roots of the equation.

250. If the number only of the real roots is required, it will be sufficient to substitute $+\infty$, and $-\infty$, the extreme limits, instead of x.

Ex. 1. Required the number of real roots in the equation $x^3-9x^2+23x-15=0$.

We shall have for the functions

$$\begin{aligned} V &= x^3 - 9x^2 + 23x - 15 \\ V_1 &= 3x^2 - 18x + 23 \\ V_2 &= x - 3 \\ V_3 &= 4. \end{aligned}$$

Substituting in these functions $+\infty$ and $-\infty$ successively, we have the following results:

$x=+\infty$ gives $+ + + +$ No variations.
$x=-\infty$ " $- + - +$ 3 variations.

Here the difference of the variations is 3. There will be, therefore, three real roots in the proposed equation. And as the equation contains no permanence, the three roots will all be positive.

If the situation as well as number of the roots is required, we proceed as follows. Beginning with 0, we substitute the series of natural numbers 1, 2, 3, 4, &c., for x in the functions, thus,

		V	V_1	V_2	V_3	
$x=0$	gives	$-$	$+$	$-$	$+$	3 variations.
$x=1$	"	0	$+$	$-$	$+$	2 "
$x=2$	"	$+$	$-$	$-$	$+$	2 "
$x=3$	"	0	$-$	0	$+$	1 variation.
$x=4$	"	$-$	$-$	$+$	$+$	1 "
$x=5$	"	0	$+$	$+$	$+$	no variation.

Since the numbers 1, 3, 5, reduce the proposed to 0, these are the roots of the equation, which are thus completely determined by the process.

Ex. 2. To find the number and initial figures of the roots of the equation $x^3 - 4x^2 - 6x + 8 = 0$.

The functions are
$$\begin{aligned} V &= x^3 - 4x^2 - 6x + 8 \\ V_1 &= 3x^2 - 8x - 6 \\ V_2 &= 17x - 12 \\ V_3 &= +. \end{aligned}$$

The sign only of V_3 is written, since this is all that is necessary, and this may be determined without actually performing the division.

Substituting $+\infty$, and $-\infty$,

$x = +\infty$ gives $+ + + +$ no variations.
$x = -\infty$ " $- + - +$ 3 variations.

There will be, therefore, three real roots; and as there is one permanence in the proposed, one of these will be negative.

To determine the situation of the roots, we substitute the series of natural numbers 0, 1, 2, &c., positive and negative, thus:

	V	V_1	V_2	V_3	Var.
$x = 0$ gives	+	−	−	+	2
$x = 1$ "	−	−	+	+	1
$x = 2$ "	−	−	+	+	1
$x = 3$ "	−	−	+	+	1
$x = 4$ "	−	+	+	+	1
$x = 5$ "	+	+	+	+	0

	V	V_1	V_2	V_3	Var.
$x = 0$ gives	+	−	−	+	2
$x = -1$ "	+	+	−	+	2
$x = -2$ "	−	+	−	+	3

From the column of variations, it is evident that the roots lie between 0 and 1, 4 and 5, -1 and -2. The initial figures of the roots are, therefore, 0, 4, and -1. To obtain the first decimal figure of the root between 0 and 1, we substitute in the order of tenths. And since 1 is very near the root we begin with .9. Thus

$x = 1$ gives $- - + +$ 1 variation.
$x = .9$ " $+ - + +$ 2 variations.

The initial figures of the roots are, therefore, .9, 4 and -1.

Ex. 3. How many real roots has the equation $x^3 - 6x^2 + 11x - 6 = 0$? Ans. Three, viz., 1, 2, and 3.

Ex. 4. How many real roots has the equation $x^3-5x^2+8x-1=0$? Ans. 1.

Ex. 5. How many real roots has the equation $x^3-7x+7=0$?

Ans. 3. Two between 1 and 2, and one between -3 and -4.

SECTION XXVII. — Solution of numerical equations of any degree.

The principles now obtained are sufficient for the solution of numerical equations of any degree with one unknown quantity. We proceed to apply them.

COMMENSURABLE ROOTS.

251. A commensurable root, it will be recollected, is one which has a common measure with unity. It may, therefore, be an integer, or a definite fraction, such as, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, &c. We commence with the research for integral roots.

Since every root of an equation (art. 229) is a divisor of the last or absolute term, the integral roots of equations must, as we have seen, be sought among the entire divisors of this term. They will be comprised, moreover, between the limits of the roots. Every equation, the coefficients of which are entire numbers and that of the first term equal to unity must have (art. 230) entire numbers for its commensurable roots. We begin with equations of this description.

Ex. 1. Let it then be proposed to find the integral roots of the equation $x^5-5x^4+x^3+16x^2-20x+16=0$.

The limits of the roots are 5 and -4. The integral roots of the equation must, therefore, be found among the entire divisors of 16 comprised between 5 and -4. These are 4, 2, 1, -1, -2, -4. But 1 and -1, it will be seen at once, are

not roots, since they will not satisfy the equation. The trial divisors, are, therefore, reduced to 4, 2, -2, -4. These we try in succession.

$$\begin{array}{r|l}
1 & 1-5+1+16-20+16 \\
4 & \underline{+4-4-12+16-16} \\
 & -1-3+4-4 \\
2 & \underline{2+2-2+4} \\
 & 1-1+2 \\
-2 & \underline{-2+2-2} \\
 & 1-1+1.
\end{array}$$

From the operation, 4, it is evident, is a root, since the division by $x-4$ leaves no remainder. Dividing the quotient of this division by $x-2$, there is no remainder; 2 is, therefore, a root. Dividing next this last quotient by $x+2$, there is no remainder; hence -2 is a root. We proceed no further with the negative divisors, since the equation exhibiting but one permanence can have but one negative root, and that is now found. The coefficients of the remainder after these successive divisions are $1-1+1$. Hence we shall have for the depressed equation containing the remaining roots of the proposed, $x^2-x+1=0$, the roots of which are imaginary.

Ex. 2. What are the integral roots of the equation

$$x^3-10x^2+31x-30=0\,?$$

Since the equation exhibits only variations of sign there can be no negative roots. It will be necessary, therefore, to find only the superior limit of the positive roots, and to employ the divisors of 30 between 0 and this limit. The limit is 10; and since 1 is obviously not a root, the trial divisors will be 2, 3, 5, and 6, by means of which we obtain 2, 3, and 5, for the roots. And the equation being of the third degree only, these are all the roots, and the equation is completely solved.

Ex. 3. What are the integral roots of the equation $x^4-10x^3+37x^2-60x+36=0$?

The equation has no negative roots. Applying the process we obtain 2 and 3 for the positive roots. These, it is evident,

may be equal roots. To discover whether this is the case, we continue the division by 2 and 3 upon the depressed equation after taking out the roots 2 and 3. Or we substitute 2 and 3 successively in the derivative of the proposed. And since they satisfy this equation also, they are (art. 247) equal roots. Thus the integral roots of the proposed are 2, 2, 3, 3; and since it is of the fourth degree only, these are all its roots.

252. When the number of trial divisors is large, the process is laborious. It will admit of some simplification in the manner we will now explain.

Let it be proposed, for example, to find the integral roots of the equation $x^3 - 15x^2 + 74x - 120 = 0$.

The equation has no negative roots. The superior limit to the positive roots is 15; and since 1 is obviously not a root, the trial divisors will be 12, 10, 8, 6, 5, 4, 3, 2.

In making trial of the divisors it will be found most convenient to invert the order of the coefficients, that is, arrange with reference to the ascending powers of x, and then change the signs of the divisor, the effect of which will be merely to change the signs of the quotient.

Thus, in the present example, if we begin with 3, the work will be

$$\begin{array}{r|l} 3 & -120 + 74 - 15 + 1 \\ 1 & \underline{\ -40} \\ & \ \ 34. \end{array}$$

3 is not a root, since 34 is not divisible by 3, and the total division by 3 is, therefore, impossible. Try next 4.

$$\begin{array}{r|l} 4 & -120 + 74 - 15 + 1 \\ 1 & \underline{ - 30 + 11 - 1} \\ & \ 44 - \ 4 \quad 0. \end{array}$$

4 is a root, since there is no remainder. Proceeding with the process we find 5 and 6 for the remaining roots.

If the operations performed above be examined with attention, the following principle, which may easily be proved to be general, will be discovered, viz.: In order that a number may be a root, the first coefficient or absolute term, must be divisible

by it, so must also the sum of the quotient and the next coefficient, the sum of this last quotient and the next coefficient, and so on throughout, the last quotient being always -1. Any number which will not sustain these tests in succession is not a root.

Taking advantage of what has now been said, the entire work in the example proposed may be conveniently performed as follows:

-120	12,	10,	8,	6,	5,	4,	3,	2,
$+74$	$-10,$	$-12,$	$-15,$	$-20,$	$-24,$	$-30,$	$-40,$	$-60,$
	64,	62,	59,	54,	50,	44,	34,	14,
-15	*	*	*	9,	10,	11,	*	7
				$-6,$	$-5,$	$-4,$		-8
$+1$				$-1,$	$-1,$	$-1,$		-4
								-3
								*

The coefficients of the proposed are placed in a vertical column with their proper signs. On the same horizontal line with the first coefficient, or absolute term, are arranged the trial divisors. Beneath these, in the same horizontal line with the second coefficient, are placed the quotients arising from the division of the absolute term by the trial divisors. Beneath these, in the next line, are placed the sums of the first quotient and second coefficients. These are the dividends to be divided next by the trial divisors; and the quotients of this division are placed beneath them in the next line below, and in the same line with the third coefficient, and so on in order.

On the second division the divisors 12, 10, 8, and 3 are rejected, and the divisor 2 on the fourth. The only divisors which sustain the required tests are 6, 5, and 4, and these are, therefore, roots of the equation.

The following examples will serve as an additional exercise.

Ex. 1. What are the integral roots of the equation $x^3+3x^2-8x+10=0$? Ans. -5.

Ex. 2. What are the integral roots of the equation $x^4+4x^3-x^2-16x-12=0$? Ans. $2, -1, -2$, and -3.

Ex. 3. What are the integral roots of the equation $x^4 - x^3 - 13x^2 + 16x - 48 = 0$? Ans. 4, and -4.

Ex. 4. What are the integral roots of the equation $x^4 - 7x^3 + 17x^2 - 17x + 6 = 0$? Ans. 1, 1, 2, and 3.

253. The method above is applicable also to equations, the coefficients of which are entire numbers and that of the first term different from unity. In this case the last quotient will be -1 multiplied by the coefficient of the first term.

Ex. 1. Find the integral roots of the equation $2x^3 - 22x^2 + 62x - 42 = 0$. Ans. 1, 3, and 7.

Ex. 2. What are the integral roots of the equation $3x^3 - 23x^2 + 44x - 20 = 0$? Ans. 2 and 5.

254. We proceed next to equations which contain fractional commensurable roots.

Let it be proposed, for example, to find the commensurable roots of the equation

$$x^4 - \frac{37}{12}x^3 + \frac{61}{24}x^2 - \frac{19}{24}x + \frac{1}{12} = 0. \quad (1)$$

To find the integral roots, we free the equation from denominators, which will not alter the value of the roots. This gives

$$24x^4 - 74x^3 + 61x^2 - 19x + 2 = 0.$$

Applying next the process above to this equation we obtain 2 for the integral root. Removing this from the proposed, the reduced equation will be

$$x^3 - \frac{13}{12}x^2 + \frac{3}{8}x - \frac{1}{24} = 0. \quad (2)$$

Since 2 is the only integral root of the proposed, the commensurable roots of this last, if it have any, must be fractional. To determine these we transform the equation into another whose roots shall be integral. In order to this (art. 240, no. 2) we put $x = \frac{x}{24}$, 24 being the least common multiple of the denominators, and we have for the transformed

$$x^3 - 26x^2 + 216x - 576 = 0. \quad (3)$$

Applying the process for integral roots to this equation the

roots are found to be 12, 8, and 6. But these, it is evident, are 24 times larger than those of equation (2). Hence the roots of (2) are $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$.

The roots of the proposed equation will be, therefore, 2, $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$.

Ex. 2. Find the commensurable roots of the equation $x^3 - \frac{31}{30}x^2 + \frac{1}{3}x - \frac{1}{30} = 0$. Ans. $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{5}$.

Ex. 3. Find the commensurable roots of the equation $3x^3 - 14x^2 + 21x - 10 = 0$. Ans. 1, $\frac{5}{3}$, and 2.

Ex. 4. What are the commensurable roots of the equation $12x^4 + 20x^3 - 11x^2 - 5x + 2 = 0$?

Ans. $\frac{1}{2}$, $\frac{1}{3}$, $-\frac{1}{2}$, and -2.

INCOMMENSURABLE ROOTS.

255. Incommensurable Roots are those which have no common measure with unity, such as surds and interminable decimals.

We proceed next to equations having incommensurable roots. These may be found by the principles explained above to any degree of approximation we please.

Let it be proposed, for example, to find the roots of the equation $x^3 + 11x^2 - 102x + 181 = 0$.

The substitution of $+\infty$ and $-\infty$ shows that the equation has three real roots. And since there is but one permanence, two of the roots will be positive. The functions are

$$\begin{aligned} V &= x^3 + 11x^2 - 102x + 181 \\ V_1 &= 3x^2 + 22x - 102 \\ V_2 &= 122x - 393 \\ V_3 &= + \end{aligned}$$

Putting $x = 0$	gives	$+ - - +$	2 variations.
$x = 1$	"	$+ - - +$	
$x = 2$	"	$+ - - +$	
$x = 3$	"	$+ - - +$	2 "
$x = 4$	"	$+ + + +$	no variation.

The two positive roots are, therefore, comprised between 3 and 4. To approach them more nearly, we transform the proposed equation into another whose roots shall be less by 3. The functions will be,

$$\begin{aligned}
V &= x^3 + 20x^2 - 9x + 1 \\
V_1 &= 3x^2 + 40x - 9 \\
V_2 &= 122x - 27 \\
V_3 &= +.
\end{aligned}$$

Putting in these

$x = 0$	gives	$+ - - +$	2 variations.
$x = .1$	"	$+ - - +$	
$x = .2$	"	$+ - - +$	2 "
$x = .3$	"	$+ + + +$	no variation.

The two positive roots of the reduced equation are comprised, therefore, between .2 and .3. Hence those of the proposed are comprised between 3.2 and 3.3.

To approach the roots still nearer we transform the last transformed equation, $x^3 + 20x^2 - 9x + 1$, into another, the roots of which shall be less by .2. The functions will then be

$$\begin{aligned}
V &= x^3 + 20.6x^2 - .88x + .008 \\
V_1 &= 3x^2 + 41.2x - .88 \\
V_2 &= 122x - 2.6 \\
V_3 &= +.
\end{aligned}$$

In these

$x = 0$	gives	$+ - - +$	2 variations.
$x = .01$	"	$+ - - +$	2 "
$x = .02$	"	$- - - +$	1 variation.
$x = .03$	"	$+ + + +$	no variation.

The two positive roots of the last transformed are comprised, therefore, between .01, .02 and .02, .03. By consequence the first three figures in those of the proposed are 3.21, and 3.22; and since the sum of the three roots is -11, the negative root will be -17.43. We shall have, therefore, for the approximate roots of the proposed 3.21, 3.22, and -17.43.

By the process above we may approach the roots of an equation as nearly as we please. The process is, however, laborious,

and may be much abridged. The following method of accomplishing this object was first published by W. G. Horner, of Bath, England, in 1819.

HORNER'S METHOD OF APPROXIMATION.

256. This method is founded upon the following principles. Let there be the equation

$$V = x^n + A x^{n-1} + \ldots\ldots T x + U = 0.$$

Let x' be the part of the root of this equation already found, and y the remaining part, y being very small compared with x'. Then, transforming the proposed into another, the roots of which shall be less by x', we obtain

$$V' = y^n + A' y^{n-1} + \ldots\ldots T' y + U' = 0;$$

then, since y is a very small quantity, all the terms of this equation, in which the power of y is above the first, may be neglected, and the equation $T' y + U' = 0$, will give the value of y very nearly. Resolving this equation we have

$$y = -\frac{U'}{T'}.$$

Thus, in the equation $x^3 - 5x^2 + 8x - 1 = 0$, the value of x, it is easy to see, lies between 0 and 1. Neglecting the terms which involve x above the first power, we have for a trial equation $8x - 1 = 0$, from which we obtain $x = .125$. The first figure of this decimal is true, since by substitution it will be found that the value of x lies between .1 and .2. Thus the first figure of the approximate root is easily determined. In order to proceed to the next, we transform the equation $V' = 0$ into another whose roots shall be less by the part last found. This will give a new trial equation, by which to find the second figure of the root, and so on. The rule may be thus stated:

1°. Find by trial, or by Sturm's Theorem, the situation and first figure of the real roots.

2°. Transform the proposed equation into another, whose roots shall be less than those of the given equation by the part of the root already found.

3°. With the absolute term in this transformed equation for a dividend, and the coefficient of x for a divisor, find the next figure of the root and verify it in the transformed equation.

4°. Diminish the roots of the transformed equation by the value of the figure last obtained, divide as before for the next figure, and so on.

Ex. 1. Find the roots of the equation $x^3+10x^2+5x-260=0$. The functions will be

$$\begin{aligned} V &= x^3+10x^2+5x-260 \\ V_1 &= 3x^2+20x+5 \\ V_2 &= 17x+239 \\ V_3 &= -. \end{aligned}$$

Substituting in these functions as above, the equation, it will be found, has but one real root, the first figure of which is 4. Transforming it into another, the roots of which shall be less by 4, the operation will be

$$\begin{array}{r|rrrr} 1 & 1 & +10 & +5 & -260 \\ 4 & & \underline{4} & \underline{56} & \underline{244} \\ & & 14 & 61 & -16 \\ & & \underline{4} & \underline{72} & \\ & & 18 & 133 & \\ & & \underline{4} & & \\ & & 22 & & \end{array}$$

and the transformed will be $x^3+22x^2+133x-16=0$. (2)

We have, therefore, for the first trial equation $133x-16=0$, which gives .1 for the second figure of the root. Thus the root of the proposed will be 4.1 nearly. Transforming next the equation (2) into another, whose roots shall be less by .1, the operation will be

$$\begin{array}{r|rrrr} 1 & 1 & +22. & +133 & -16 \\ .1 & & \underline{0.1} & \underline{2.21} & \underline{13.521} \\ & & 22.1 & 135.21 & -2.479 \\ & & \underline{.1} & \underline{2.22} & \\ & & 22.2 & 137.43 & \\ & & \underline{.1} & & \\ & & 22.3 & & \end{array}$$

and the transformed will be $x^3 + 22.3\,x^2 + 137.43\,x - 2.479 = 0.$ (3)

We have, therefore, for the next trial equation, $137.43\,x - 2.479 = 0$, which gives .01 for the next figure of the root, The root will be, therefore, 4.11 nearly. Transforming, next, equation (3) into another, whose roots shall be less by .01, the operation will be

$$\begin{array}{r|l r r r}
1 & 1 & +22.3 & +137.43 & -2.479 \\
.01 & & .01 & .2231 & 1.376531 \\ \cline{3-5}
 & & 22.31 & 137.6531 & -1.102469 \\
 & & .01 & .2232 & \\ \cline{3-4}
 & & 22.32 & 137.8763 & \\
 & & .01 & & \\ \cline{3-3}
 & & 22.33 & &
\end{array}$$

and the transformed will be

$$x^3 + 22.33\,x^2 + 137.8763\,x - 1.102469 = 0. \quad (4)$$

And we have for the next trial equation $137.8763\,x - 1.102469 = 0$, by which we obtain .007 for the next figure of the root. Transforming (4) into another, whose roots shall be less by .007, the operation will be

$$\begin{array}{r|l r r r}
1 & 1 & +22.33 & +137.8763 & -1.102469 \\
.007 & & .007 & .156359 & .966228613 \\ \cline{3-5}
 & & 22.337 & 138.032659 & -.136240387 \\
 & & .007 & .156408 & \\ \cline{3-4}
 & & 22.344 & 138.189067 & \\
 & & .007 & & \\ \cline{3-3}
 & & 22.351 & &
\end{array}$$

and the transformed will be

$$x^3 + 22.351\,x^2 + 138.189067\,x - .136240387 = 0, \quad (5)$$

and the next trial equation is $138.189067\,x - .136240387 = 0$, from which we obtain .0009 for the next figure of the root. The root of the proposed will be, therefore, 4.1179 nearly. In like manner the approximation may be pushed as far as we please.

The calculations may be performed more concisely as in the following table:

1	1	+ 10	+ 5	− 260	4.1179 +
4		4	56	244	
		14	61	* − 16	
		4	72	13.521	
		18	* 133	* − 2.479	
		4	2.21	1.376531	
1.		* 22.1	135.21	* − 1.102469	
.1		.1	2.22	.966228613	
		22.2	* 137.43	* − .136240387	
		.1	.2231		
1		* 22.31	137.6531		
.01		.01	.2232		
		22.32	* 137.8763		
		.01	.156359		
1		* 22.337	138.032659		
.007		7	.156408		
		22.344	* 138.189067		
		7			
		* 22.351			

The process, when compared with the previous work, will be easily understood. The coefficients of the successive transformed equations are marked with a star. The .1 placed at the right of * 22 in the first column is the .1 added in the first step of the process for obtaining the second transformed equation, the addition being more conveniently made in this manner. A similar remark applies to the right hand figure of the other coefficients in the same column.

The successive figures may be verified as they are obtained in the transformed equation. The process, it is evident, becomes more accurate as we proceed. After four or more decimals have been obtained, two or three more may in general be found by simple division.

The sign of the last term will sometimes change in the course of the operation. Unless there is in this case a change

of sign in the preceding column also, the figure which has given rise to the change must be incorrect. This change may not, however, always occur at the same figure of the root.

257. In the preceding example a greater number of decimal places has been employed than is necessary to obtain the root to the degree of approximation attained, and the work might have been abridged by the omission of some of them.

Let it be required, as a second example, to find the roots of the equation $x^3 - 17x^2 + 54x - 350 = 0$, to three places of decimals.

The equation has but one real root, the first two figures of which, found by trial, or Sturm's Theorem, are 14. The remainder of the work, according to the rule, will be as follows:

```
   1 | 1     — 17         + 54           — 350 | 14.954
  14 |         14         — 42             168
           ———————      ———————       ——————————
           —   3            12         * — 182
              14           154             170.379
           ———————      ———————       ——————————
              11         * 166         * —  11.621 |
              14            23.31           10.740 | 875
           ———————      ———————            ——————————————
   1 |     * 25.9          189.31        * — 880 | 125
  .9 |        .9            24.12            865 | 275664
           ———————      ———————            ——————————————
             26.8        * 213.4|3          — 14 | 849336
               .9            1.3|875
           ———————      ————————————
   1   |   * 2|7.75        214.8|175
  .05  |          5          1.3|900
           ———————      ————————————
             27.80       * 216.|2075
                 5             |111416
           ———————      ————————————
   1.   |  * 27.854        216.|318916
  .004  |        4             |111432
           ———————      ————————————
             27.858        216.|430348
                  4
           ———————
             27.862
```

If this work is examined with attention, it will be seen that the result will still remain the same, if all the figures to the right of the vertical lines, and those below the vertical line in

the left hand column are omitted, by which the labor will be much abridged.

The object is to retain no more decimal places in the last column than are necessary; and these, in general, will not be greater than the number of places required in the root. Thus, in the present example, three places only being required in the root, we cut off in the last column, by a vertical line, all the remaining figures after three places have been obtained. This occurs after the operations with the figure 9 of the root have been completed. Then, since the multiplication by each new figure of the root produces one new decimal place in each column as we proceed from left to right, in order that no new decimal places may occur in the right hand column we must, it is evident, cut off one figure in the column next preceding, two figures in the column next preceding that, and so on. The operation with the figure 5 of the root being completed, we again cut off one figure in the column next preceding the last, and two in the column next preceding that, and so on. The work will then stand thus:

1	− 17	54	− 350	14.95407
14	14	− 42	168	
	− 3	12	* − 182	
	14	154	170.379	
	11	* 166	* − 11.621	
	14	23.31	10.741	
1	* 25.9	189.31	− .880	
.9	9	24.12	865	
	26.8	* 213.4|3	− 15	
	9	1.3|8	15	
1	* 2|7.7	214.8|1		
.05	| 1	1.3|9		
1				
004	|··| .2|7.8	2|1|6|.2		

Care, it is evident, must be taken that, in the result of each operation, the figure immediately preceding the one cut off remain the same as if the contraction were not made. Thus

in the operation with the figure 5 of the root, we continue the use of the 77, cut off in the left hand column, until the operation with the 5 is completed. We retain also, in the column next following, one of the decimal places cut off. This gives, when the operation with the 5 is completed, 216.2 in this column, precisely as if the contraction were not made. Cutting off next one figure, the 2, in the column before the last, and two in the column before that, there will be none remaining in this column, and the multiplication by 4, the next figure of the root, will produce no effect upon the following columns.

Having obtained the 4, two additional figures may be found by simple division. In order to this, cutting off the 6 in the column before the last, we have 21 for a divisor and 15 for a dividend, which gives 0 for the next figure of the root. Again cutting off the 1 in the column before the last, we have 2 for a divisor and 15 for a dividend, which gives 7 for the next figure of the root; and the operation, true to 5 places of decimals, is now terminated.

There is room for the exercise of judgment in the use of the figures cut off. The object being to keep the right hand figure in the result of each operation what it would be if the contraction were not made, the learner must judge, in each case, what is necessary for this purpose. What has been done will serve as a general direction. After a little practice the operations will be easily executed, and the root obtained, to any degree of approximation required, with extreme facility.

Ex. 3. To find the roots of the equation $x^3 - 7x + 7 = 0$. There will be one negative and two positive roots. To find the negative root, we change the sign of the alternate terms, and proceed as for a positive root. The result, with its sign changed, will be the negative root sought.

The roots are 1.356895, 1.692021, and -3.048917, true to six places of decimals.

Two of the roots being obtained, the other may be found by

the principle, art. 229. Thus, to find the negative root, we take the sum of the two positive roots with the contrary sign.

Ex. 4. Find one root of the equation $x^3 + 3x^2 + 5x - 178 = 0$, true to 6 places of decimals. Ans. 4.538825.

Ex. 5. Find the roots of the equation $x^3 - 5x - 3 = 0$, true to four places of decimals.

Ans. 2.4908, $-$ 0.6566, $-$ 1.8342.

Ex. 6. Find a root of the equation $x^3 + 2x^2 - 23x - 70 = 0$, true to four places of decimals. Ans. 5.1345.

Ex. 7. Find the root of the equation $x^5 + 3x^4 + 2x^3 - 3x^2 - 2x - 2 = 0$, to four places of decimals. Ans. 1.0591.

Ex. 8. Find the roots of the equation $x^4 - x^3 + 2x^2 + x - 4 = 0$. Ans. 1.14699459, and $-$ 1.0905935.

258. The preceding process may be applied to the extraction of the roots of numbers.

Ex. 1. Let it be required to extract the third root of 9.

In this case, we have to solve the equation $x^3 - 9 = 0$, which, by the preceding process, gives 2.0800838 for the answer.

Ex. 2. To find the roots of the equation $x^2 - 2 = 0$.

Ans. $\pm$ 1.414213.

Ex. 3. Find the fifth root of 2. Ans. 1.148699.

SECTION XXVIII. Elimination. — Solution of Equations with Two or More Unknown Quantities.

259. The equations of any degree, thus far solved, contain one unknown quantity only. We proceed next to equations with more than one unknown quantity.

Let there be two equations with two unknown quantities. It is proposed to find the systems of values for the unknown quantities x and y, that will satisfy these equations. In order to this, we must first eliminate one of the unknown quantities.

Let the equations, arranged in reference to x, be represented by

$$A = 0, B = 0.$$

Applying to these the principle of the greatest common divisor, let $Q =$ the quotient of A by B, and $R =$ the remainder; then

$$A = BQ + R.$$

It follows from this equality, that all the values of x and y which give $A = 0$, $B = 0$, must also give $R = 0$. The system of equations $A = 0$, $B = 0$, may, therefore, be replaced by the more simple system, $B = 0$, $R = 0$.

Dividing next B by R, let a new remainder R' be reached. We may, in like manner, substitute for $B = 0$, $R = 0$, the system $R = 0$, $R' = 0$, in which R' is of a lower degree in respect to x than R. And we may thus continue until a remainder, R'' for example, is obtained independent of x. The original system, $A = 0$, $B = 0$, may then be replaced by the system $R' = 0$, $R'' = 0$, in which R'' contains y only, and R' is generally of the first degree in x. The equation $R'' = 0$, from which x is eliminated, is called the *final* equation.

The values of y being obtained from the final equation, those of x will be easily found; and thus the systems of values for x and y, proper to satisfy the proposed equations, be determined.

Ex. 1. To find the values of x and y in the equations

$$x^2 + y^2 - 34 = 0$$
$$x + y - 8 = 0.$$

Applying the process of the greatest common divisor,

$$\begin{array}{r|l}
x^2 + \qquad\qquad y^2 - 34 & x + y - 8 \\
\underline{x^2 + (y-8)\,x \qquad\qquad} & \overline{x - y + 8} \\
-\,(y-8)\,x - y^2 - 34 & \\
\underline{-\,(y-8)\,x - y^2 + 16\,y - 64} & \\
2\,y^2 - 16\,y + 30 = \text{Remainder.} &
\end{array}$$

Putting this remainder equal to 0, we have for the final equation,

$$y^2 - 8\,y + 15 = 0.$$

Resolving this equation we obtain $y = 3$ or 5; and, substituting in the divisor, we obtain for the corresponding values of x, $x = 5$ or 3. Thus the system of values for x and y, which satisfy the proposed equations, are $y = 3$, $x = 5$, and $y = 5$, $x = 3$.

Ex. 2. Find the values of x and y in the equations,

$$x^2 y^4 + y^2 - 333 = 0,$$
$$x\, y^2 + y - 21 = 0.$$

Ans. $y = 3$, $x = 2$, or $y = 18$, $x = \frac{1}{108}$.

Ex. 3. Find the values of x and y in the equations

$$4x - 2y + y^2 - 11 = 0,$$
$$x + 4y - 14 = 0.$$

Ans. $y = 3$, $x = 2$, or $y = 15$, $x = -46$.

260. In the process for finding the greatest common divisor, it may be necessary, in order that the division may be exactly performed, to multiply one of the quantities by a factor containing y. In this way roots may be introduced into the final equation foreign to the proposed equations, and which must be rejected.

In like manner, for convenience, factors containing y may be suppressed in the course of the operations, which, when put equal to 0, may give values for y proper to satisfy the equations, but which, from the suppression of the factors, will not be found in the final equation. We must, therefore, in order to a complete solution, make the factors introduced or suppressed equal 0. We shall thus establish relations which, with the final equation, will enable us readily to detect the foreign solutions and to determine all the values of the unknown quantities proper to satisfy the proposed equations.

261. Various simplifications may be introduced into the operations, and the process improved so as to avoid the foreign solutions. The general idea of the process we have given is all our limits admit. We subjoin a few additional examples.

Ex. 1. To find the values of x and y in the equations,

$$x^2 - y^2 - 6x - 9 = 0,$$
$$x^2 + 2xy + y^2 - 1 = 0.$$

Ans. $y = -1$, $x = 2$, and $y = -2$, $x = 1$.

Ex. 2. To find the values of x and y in the equations,

$$x^3 - 3yx^2 + (3y^2 - y + 1)x - y^3 + y^2 - 2y = 0,$$
$$x^2 - 2yx + y^2 - y = 0. \qquad \textit{Ans.}\ x = 2,\ 7 = 1.$$

Ex. 3. To find the values of x and y in the equations,

$$y x^3 - (y^3 - 3y - 1)x + y = 0,$$
$$x^2 - y^2 + 3 = 0.$$

Ans. The equations are incompatible.

It will be easy to see how we are to proceed with three equations with three unknown quantities, and so on.

SECTION XXIX. Infinite Series.

262. An infinite series is one in which the number of terms is unlimited; the law of the series being generally discoverable by an examination of a few of the terms.

A *converging* series is one whose successive terms decrease, or become less and less.

A *diverging* series is one whose successive terms increase, or become greater and greater.

An *ascending* series is one in which the exponents of the unknown quantity continually increase; and a *descending* series is one in which the exponents continually decrease.

When the value of an algebraic expression cannot be exactly determined, we expand the expression into a series, and thus endeavor to obtain an approximate value. We have, therefore, two questions to solve in respect to series. 1°. To expand algebraic expressions into series. 2°. To find any term of a series, and the sum of all the terms.

UNDETERMINED COEFFICIENTS.

263. In the development of algebraic expressions in series, the method of undetermined coefficients is found of great utility.

We will give a brief exposition of this method.

Let there be the equation,

$$0 = A + Bx + Cx^2 + Dx^3 + \dots$$

in which the coefficients A, B, C . . are independent of x; it is required to determine these coefficients so that the equation

may be true whatever value is assigned to x. Since the coefficients A, B, C . . are to be determined, they are on this account called *undetermined coefficients.*

Since by hypothesis the proposed equation must be true, whatever value is assigned to x, it must be true for the particular value $x = 0$. Putting $x = 0$, the equation is reduced to $0 = A$; we have, therefore, $A = 0$. Substituting this value of A in the equation, and dividing both sides by x, it becomes

$$0 = B + C x + D x^2 + \ldots$$

but since this equation must also be true whatever the value of x, putting $x = 0$, we obtain $B = 0$. By the same course of reasoning it may be shown also that $C = 0$, $D = 0$.

If, then, we have an equation of the form

$$0 = A + B x + C x^2 + D x^3 + \&c.,$$

in which the coefficients A, B, C, &c., are independent of x, in order that the equation may be true whatever value is assigned to x, *each separate coefficient must necessarily be equal to zero.*

This is the principle upon which the method of undetermined coefficients is founded. We pass to some applications of the method.

Ex. 1. Let there be a dividend $x^2 - p x + p'$, let the divisor be $x - a$, and the quotient $x - q$; to determine the conditions necessary in order that the division may be exact.

Since, when there is no remainder, the divisor multiplied by the quotient should produce anew the dividend, we must have

$$(x - a)\ (x - q) = x^2 - p x + p',$$

or, performing the multiplication indicated, transposing and reducing,

$$0 = (a + q - p)\, x + (p' - a q);$$

but since this equation is true whatever the value of x, we must have

$$a + q - p = 0, \text{ and } p' - a q = 0;$$

eliminating q from these last, we obtain

$$a^2 = a p - p'.$$

In order, therefore, that the division may be exact, we must

have the relation $a^2 = ap - p'$, or, which is the same thing, $a^2 - ap + p' = 0$.

Ex. 2. Let it be proposed to decompose $\dfrac{3+5x}{x^2-5x+6}$ into fractions, whose sum is the given fraction, and whose denominators are the factors of the given denominator.

The factors of the denominator are $x-3, x-2$; we assume, therefore,

$$\frac{3+5x}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}.$$

Freeing from denominators, transposing and reducing

$$0 = (A + B - 5)\,x - (2A + 3B + 3;$$

whence $\quad A + B = 5$, and $2A + 3B = -3$;

from which we obtain $A = 18$, $B = -13$, and we have

$$\frac{3+5x}{x^2-5x+6} = \frac{18}{x-3} - \frac{13}{x-2}.$$

Ex. 3. Find for A and B values, such that we may have

$$\frac{7+9x}{(x-5)(x-3)} = \frac{A}{x-5} + \frac{B}{x-3}.$$

Ans. $A = 26$, $B = -17$.

Ex. 4. Find for A and B values such that we may have

$$\frac{3x-5}{(x-4)(x-2)} = \frac{A}{x-4} - \frac{B}{x-2}.$$

Ans. $A = \frac{7}{2}$, $B = \frac{1}{2}$.

264. We proceed to the application of the method to the development of algebraic expressions in series.

Ex. 1. Let it be proposed, to develop the expression $\dfrac{z}{x+z}$ in series, according to the ascending powers of x. In order to this, we assume

$$\frac{z}{z+x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots$$

the coefficients A, B, C, being independent of x.

Freeing the first member from its denominator, transposing and arranging with reference to x, we obtain

$$0 = \begin{array}{c} A\,z \\ -z \end{array} \left| x^0 + \begin{array}{c} B\,z \\ A \end{array} \right| x^1 + \begin{array}{c} C\,z \\ B \end{array} \left| x^2 + \begin{array}{c} D\,z \\ C \end{array} \right| x^3 + \ldots$$

we have, therefore, the series of equations

$A\,z - z = 0$, $B\,z + A = 0$, $C\,z + B = 0$, $D\,z + C = 0$, &c.

from which to deduce the values of A, B, C, &c. Performing the operations and substituting in the assumed expression, we obtain for the development required

$$\frac{z}{z+x} = 1 - \frac{x}{z} + \frac{x^2}{z^2} - \frac{x^3}{z^3} + \frac{x^4}{z^4} - \ldots$$

Ex. 2. Expand the fraction $\frac{1}{1-2x+x^2}$ into an infinite series.

Assume $\frac{1}{1-2x+x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 +$ &c.

Freeing from denominator, transposing and reducing

$$1 = A + \begin{array}{c} B \\ -2A \end{array} \left| x + \begin{array}{c} C \\ -2B \\ A \end{array} \right| x^2 + \begin{array}{c} D \\ -2C \\ B \end{array} \left| x^3 + \begin{array}{c} E \\ -2D \\ C \end{array} \right| x^4 + \text{\&c.}$$

from which we obtain $A = 1$, $B = 2$, $C = 3$, $D = 4$, &c.,

whence $\frac{1}{1-2x+x^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^5 +$ &c.

Ex. 3. Expand the fraction $\frac{1+2x}{1-x-x^2}$ into an infinite series.

Ans. $1 + 3x + 4x^2 + 7x^3 + 11x^4 + 18x^5 + 29x^6 +$, &c.

Ex. 4. Expand $\frac{1+2x}{1-3x}$ into an infinite series.

Ans. $1 + 5x + 15x^2 + 45x^3 + 135x^4 +$, &c.

What is the law of the coefficients in the three preceding series?

Ex. 5. Develop $(a-x)^{\frac{1}{2}}$ in series.

Assume $(a-x)^{\frac{1}{2}} = A + Bx + Cx^2 + Dx^3 +$ &c.

Squaring, $(a-x) = A^2 + 2ABx + \begin{array}{c} B^2 \\ 2AC \end{array} \left| x^2 + \begin{array}{c} 2BC \\ 2AD \end{array} \right| x^3 +$ &c.,

whence $A = a^{\frac{1}{2}}$, $B = -\frac{1}{2a^{\frac{1}{2}}}$, $C = -\frac{1}{2.4\,a^{\frac{3}{2}}}$ &c.,

from which the development sought will be easily obtained.

265. Let it be required next to expand $\frac{1}{3x - x^2}$ into an infinite series.

If we assume $\frac{1}{3x - x^2} = A + Bx + Cx^2 + Dx^3 +$, &c., and determine the coefficients accordingly, we shall have $-1 = 0$, $3A = 0$, $3B - A = 0$, &c., the first of which is absurd.

The proposed cannot, therefore, be developed in this form. Indeed, if we put in the proposed fraction $x = 0$, the fraction takes the form of infinity. The development, therefore, for this hypothesis, should take the same form. But in order to this it must contain, it is obvious, a term in x with a negative exponent. We assume, therefore,

$$\frac{1}{3x - x^2} = Ax^{-1} + Bx^0 + Cx + Dx^2 + Ex^3 +, \text{\&c.},$$

from which we obtain successively

$$A = \frac{1}{3}, B = \frac{1}{9}, C = \frac{1}{27}, D = \frac{1}{81}, \text{\&c.}$$

It is usual to assume the development so that it shall proceed according to the ascending powers of x, beginning with x^0. But this form will not always apply. And in any case in which it is not applicable the fact will become evident, as in the preceding example, by the appearance of some absurdity in the result of the operation.

The form which the development should take may, in general, be discovered at the outset, by putting $x = 0$ in the function to be developed, and observing the result. If the function, on the hypothesis $x = 0$, is finite, the development should be taken according to the ascending powers of x, beginning with x^0. If the function on this hypothesis becomes 0, the first term of the development should contain x. If it takes the form of infinity, the first term of the development should contain x with a negative exponent.

266. The following miscellaneous examples will serve as an additional exercise.

Ex. 1. To develop $\frac{1+x}{1-x}$ in series.

Ans. $1+2x+2x^2+2x^3+$, &c.

Ex. 2. Develop $\frac{1}{(1+x)^2}$ in series.

Ans. $1-2x+3x^2-4x^3+5x^4-$, &c.

Ex. 3. Expand $(1-x)^{\frac{1}{2}}$ into an infinite series.

Ans. $1-\frac{x}{2}-\frac{x^2}{2.4}-\frac{3x^3}{2.4.6}-\frac{3.5x^4}{2.4.6.8}-$, &c.

Ex. 4. Decompose $\frac{3x^2-1}{x(x+1)(x-1)}$ into partial fractions.

Ans. $\frac{1}{x}+\frac{1}{x+1}+\frac{1}{x-1}$.

The Differential Method.

267. Let there be any series represented by a, b, c, d, &c.; if we subtract the first term from the second, the second from the third, and so on, the differences thus obtained will form a new series called the *first order of differences.* If we subtract again, the first term from the second, the second from the third, &c., in this last series, the differences thus obtained will form a third series, called the *second order of differences*, and so on.

Thus, the series of square numbers, with the several orders of differences, is as follows:

1	4	9	16	25	36	49	
3	5	7	9	11	13		1st Difference.
2	2	2	2	2			2d "
0	0	0	0				3d "

If in a proposed series the first differences are all the same, or constant, the series is called a difference series of *the first order.* If the second differences are constant, it is called a difference series of *the second order*, and so on. Thus, the series of square numbers above is a difference series of the second order.

Ex. Of what order is the series 1, 4, 10, 20, 35, 56?

268. From what has been done we shall have the following questions to solve in respect to series: 1°. To find the successive differences of the terms of a series. 2°. By means of these to find any intermediate term, and the sum of all the terms.

1. Resuming the general series, we have

$a,$	$b,$	$c,$	$d,$	$e,$ &c.	
$b-a,$	$c-b,$	$d-c,$	$e-d$		1st Diff.
$c-2b+a,$	$d-2c+b,$	$e-2d+c$			2d "
$d-3c+3b-a,$ &c.,					3d "
. . . .					" .

If we now represent the first terms of the successive orders of differences by D_1, D_2, &c., we shall have, reversing the order of the terms,

$$\begin{aligned} D_1 &= -a+b \\ D_2 &= a-2b+c \\ D_3 &= -a+3b-3c+d \\ D_4 &= \text{\&c.} \end{aligned}$$

In which it will be perceived that the coefficients of the several terms correspond with those in the expansion of a binomial. And we shall have, generally,

$$D_n = \pm a \mp nb \pm n\frac{n-1}{1.2}c \mp n\frac{(n-1)(n-2)}{1.2.3.}d \pm \text{\&c.},$$

the upper sign corresponding to the case in which the difference is even, and the lower to the case in which it is odd.

By means of this formula, we readily find the first term in any order of differences.

Ex. 1. What is the first term of the third order of differences in the series of cubes 1, 8, 27, 64, 125, &c.?

In this case, n being odd, we use the lower signs of the formula, and we have

$$D_3 = -1 + 3.8 - 3.27 + 64 = 6.$$

Ex. 2. What is the first term of the fourth order of differences in the series 7, 12, 21, 36, 62? Ans. 3.

Ex. 3. What is the first term of the fifth order of differences in the series 1, 6, 21, 56, 126, 252, &c.? Ans. 1.

2. Let it be required next to find any intermediate term of the series.

From the expressions D_1, D_2, D_3, &c., we obtain

$$b = a + D_1$$
$$c = a + 2D_1 + D_2$$
$$d = a + 3D_1 + 3D_2 + D_3,$$

in which it will be perceived that the coefficients of the successive terms are the coefficients of the power of a binomial one degree less than the number of the term; whence, putting L for the nth term of the series

$$L = a + (n-1)D_1 + (n-1)\frac{n-2}{2}D_2 + \ldots$$

The series will terminate if the differences become 0 after a certain order. Otherwise it will be infinite, and L can be determined only approximately.

Ex. 1. What is the 50th term of the series 1, 4, 8, 13, &c.?

Here $a = 1$, $D_1 = 3$, $D_2 = 1$, $D_3 = 0$. Ans. 1324.

Ex. 2. Required the tenth term of the series 1, 4, 8, 13, 19, &c. Ans. 64.

Ex. 3. Required the twentieth term of the series 1, 5, 15, 35, 70, 126, &c. Ans. 8855.

Ex. 4. What is the twelfth term of the order of cubes, or, in other words, what is the cube of 12? Ans. 1728.

3. Let it be required next to find the sum of any number n terms of the series

$$a,\ b,\ c,\ d,\ e,\ \&c.$$

Assume the series

$$0,\ a,\ a+b,\ a+b+c,\ a+b+c+d,\ \&c.$$

Subtracting each term from the next succeeding, we obtain a, b, c, d, &c., the proposed series. From this it follows that the sum of n terms of the proposed series is the $(n+1)$th term of the assumed series, and that the nth order of differences in the first series is the same as the $(n+1)$th order in the second.

We shall have, therefore, by substituting in the formula for L above,

0 for a, $n+1$ for n, a for D_1, D_1 for D_2, &c., . . .

$$S = na + n\frac{n-1}{2}D_1 + n\frac{(n-1)(n-2)}{2.3}D_2 + \ldots$$

which is the expression for the sum of any number n terms of the series.

Ex. 1. Required the sum of 12 terms of the series

1, 4, 8, 13, 19 . . . . .

Here $a = 1$, $D_1 = 3$, $D_2 = 1$, $D_3 = 0$, $n = 12$. Ans. 430.

Ex. 2. Required the sum of n terms of the series 1, 2, 3, 4, 5, &c. Ans. $\frac{n^2+n}{2}$.

Ex. 3. Required the sum of ten terms of the series 1, 5, 15, 35, 70, 126, &c. Ans. 2002.

INTERPOLATION.

269. The formula for L may be applied to the process of interpolation, or that of finding numbers intermediate between those contained in tables.

Suppose that we have a table of the square roots of numbers, from 1 to 100, and that we wish, for example, to insert the square roots of the intermediate numbers to every $\frac{1}{4}$ of a unit. If the first differences of the roots of the numbers already in the tables were constant, we should have merely to find proportional parts of the difference between any two consecutive roots, which we should add to the roots respectively. This would be the method by first differences. But the first differences not being constant, the second must be taken into consideration, and so on, until the differences become constant, or the requisite degree of approximation has been obtained. For this purpose we employ the formula L above. Putting, for convenience, in this formula n, to represent the distance of any term from the first, we have $x = n - 1$, and the formula becomes

$$L = a + x\,D_1 + \frac{x(x-1)}{2}D_2 + \frac{x(x-1)(x-2)}{2.3}D_3 + ..$$

Let three contiguous numbers of the table, with the square roots and differences to D_2, be as follows; it is required to find the square root of $94\frac{1}{4}$, or 94.25.

No.	Square root.	D_1	D_2
94	9.69536	.05143	—.00026
95	9.74679	.05117	
76	9.79796		

In the present case we shall have $a = 9.69536$, $D_1 = .05143$, $D_2 = -.00026$, and $x = \frac{1}{4}$; whence

$$L = 9.69536 + \tfrac{1}{4}\ (.05143) + \tfrac{3}{32}\,(.00026)$$
$$= 9.70824, \quad \text{Ans.}$$

Ex. 2. Given the logarithms of 6, 7, 8, 9, and 10, to find that of 6.5.

No.	Logs.	D_1	D_2	D_3	D_4
6	0.778151	.066947	—.008955	.002116	—000673
7	0.845098	.057992	—.006839	.001443	
8	0.903090	.051153	—.005396		
9	0.954243	.045757			
10	1.000000				

$$L = a + \tfrac{1}{2}D_1 - \tfrac{1}{8}D_2 + \tfrac{1}{16}D_3 - \tfrac{5}{128}D_4 = 0.812901 \text{ Ans.}$$

Ex. 3. Given the natural sine of 37° equal to .60182
" " 38° " .61566
" " 39° " .62932
" " 40° " .64279

to find that of 37° 30′. Ans. .60876.

SUMMATION OF INFINITE SERIES.

270. The summation of an infinite series consists in finding a finite expression equivalent to the series.

Since the sum of a series must evidently depend upon the law of the series, no formula can be given for the summation of series which will apply universally. A great variety of useful series may be summed by the following principles.

Let there be a series, the terms of which shall be represented by the fraction,

$$\frac{pq}{n(n+p)}=\frac{q}{n}-\frac{q}{n+p}$$

whence
$$\frac{q}{n(n+p)}=\frac{1}{p}\left(\frac{q}{n}-\frac{q}{n+p}\right)$$

If, then, a series be represented by $\frac{q}{n(n+p)}$, its sum, it is evident, will be equal to a pth part of the difference of the two series represented by $\frac{q}{n}$ and $\frac{q}{n+p}$.

Ex. 1. What is the sum of n terms of the series

$$\frac{1}{1.2}, \ \frac{1}{2.3}, \ \frac{1}{3.4}, \ \frac{1}{4.5} \ldots ?$$

Here, in the general term $q=1$, $p=1$, $n=1, 2, 3$, &c., whence we have for the sum S,

$$\left.\begin{array}{l} S=1+\frac{1}{2}+\frac{1}{3}+\ldots\ \frac{1}{n} \\ \quad -\frac{1}{2}-\frac{1}{3}-\ldots-\frac{1}{n}-\frac{1}{n+1} \end{array}\right\}=\frac{n}{n+1}, \text{ Ans.}$$

If $n=\infty$, the sum, it is evident, will be 1.

Ex. 2. What is the sum of n terms of the series

$$\frac{1}{1.3}, \ \frac{1}{3.5}, \ \frac{1}{5.7}, \ \frac{1}{7.9}, \ \frac{1}{9.11} \ldots \text{to infinity.}$$

Here $q=1, p=2$, and $n=1, 3, 5, 7\ldots$; whence

$$S=\frac{1}{2}\left\{\begin{array}{l} 1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}\ldots\ \frac{1}{2n-1} \\ \quad -\frac{1}{3}-\frac{1}{5}-\frac{1}{7}\ldots-\frac{1}{2n-1}-\frac{1}{2n+1} \end{array}\right\}=$$

$\frac{n}{2n+1}$, Ans.

If $n=\infty$, the sum, it is evident, will be $\frac{1}{2}$.